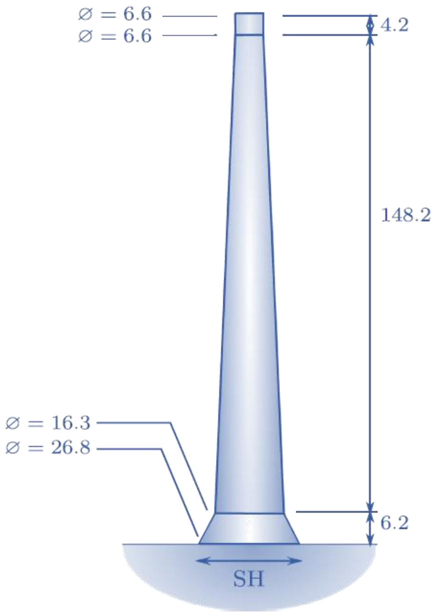
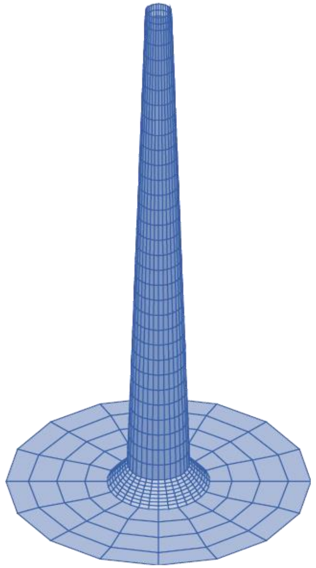
Solving Selected Nonlinear Problems with the Finite and Boundary Element Methods

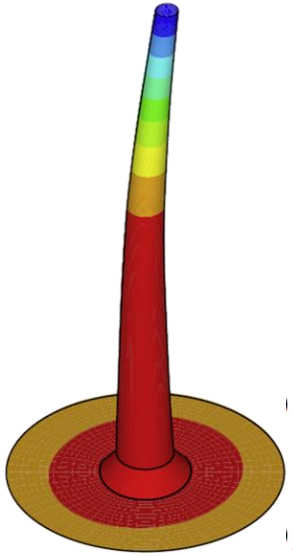
by *Prof. George D. Hatzigeorgiou*

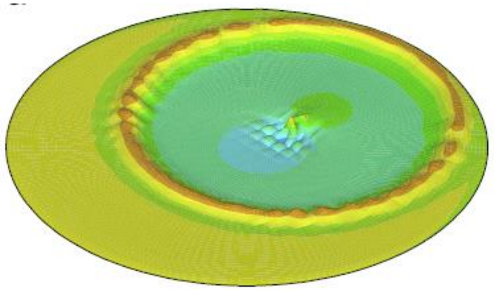
Hellenic Open University

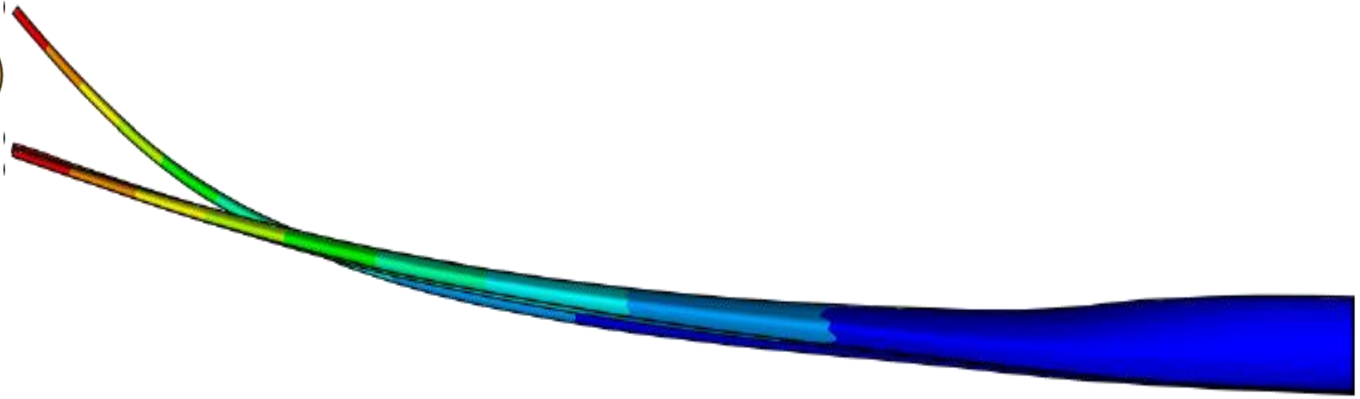
hatzigeorgiou@eap.gr











DAAD Deutscher Akademischer Austausch Dienst German Academic Exchange Service
KIT Karlsruhe Institute of Technology
UNIVERSITY OF PATRAS

Outline

- Background
- Continuum mechanics, strains and stresses
- Physical Problems in Engineering
- Sources of nonlinearity
- Linear Finite Element Methods
- Nonlinear Finite Element Methods
- Linear Boundary Element Methods
- Nonlinear Boundary Element Methods
- Conclusions

Background

A short overview of main safety and serviceability techniques for wind turbines and the related problems of computational mechanics is presented. Computational models of the leading solid mechanics and fluid mechanics problems are reviewed. For the further optimization of design and analysis technology and efficiency, multi-physical, multi-scale computational models should be employed

Continuum mechanics, strains and stresses

Continuum mechanics is a subject that unifies solid mechanics, fluid mechanics, thermodynamics, and heat transfer, all of which are core subjects of wind turbines engineering. Continuum mechanics is essentially based upon four fundamental mechanical principles, commonly known as conservation laws or balance laws:

(1) the law of conservation of mass,

(2) the law of balance of linear momentum,

(3) the law of balance of angular momentum, and

(4) the law of balance of energy.

These laws are postulated in the form of equations involving certain integrals; such equations give rise to the field equations that should hold at every point of a continuum and for all time. The important feature of the field equations is that these equations are applicable to all continua - solids, liquids, and gases - regardless of their internal physical structure.

Continuum Mechanics, strains and stresses

Axioms of Continuum Mechanics

1) A material continuum remains continuum under the action of forces.
2) Stress and strain can be defined everywhere in the body.
3) Stress at a point is related to the strain and the rate of change of strain with respect to time at the same point.
4) Stress at any point in the body depends only on the deformation in the immediate neighborhood of that point.
5) The stress-strain relationship may be considered separately, though it may be influenced by temperature, electric charge, etc.

Physical Problems in Engineering

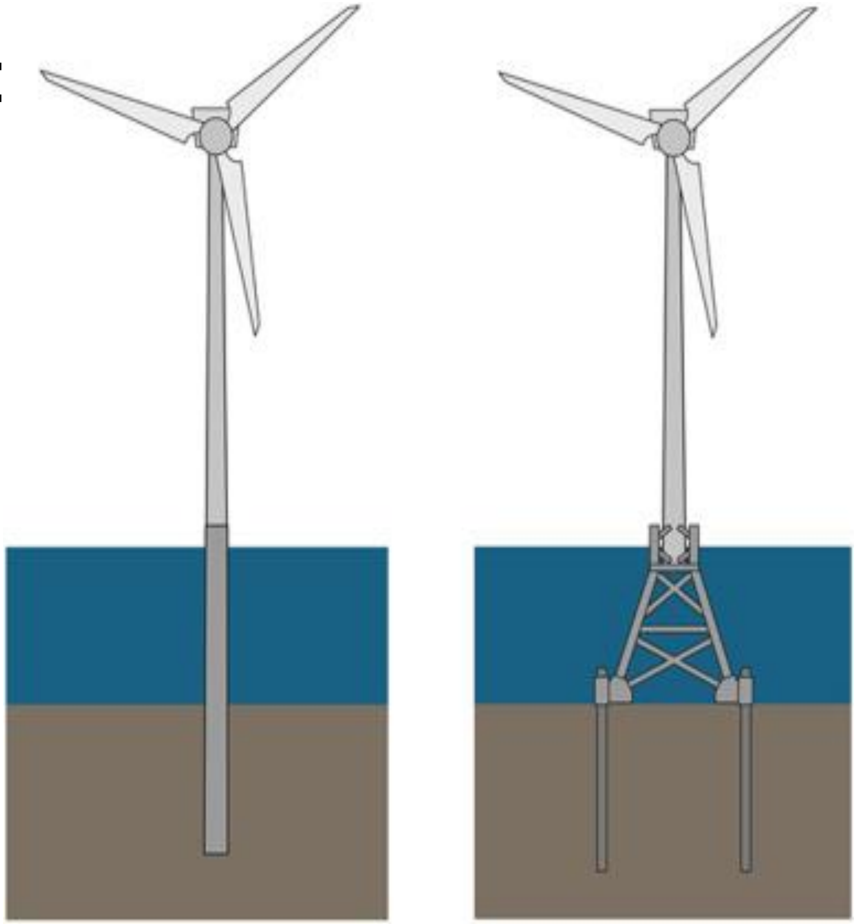
There are numerous physical engineering problems in a particular wind turbine. Common physical problems for wind turbines solved using the standard computational mechanics methods include:

- Mechanics for solids and structures.
- Heat transfer.
- Acoustics.
- Fluid mechanics.
- Others.

Physical Problems in Engineering

The procedure of computational modeling using a computational mechanics method broadly consists of four steps:

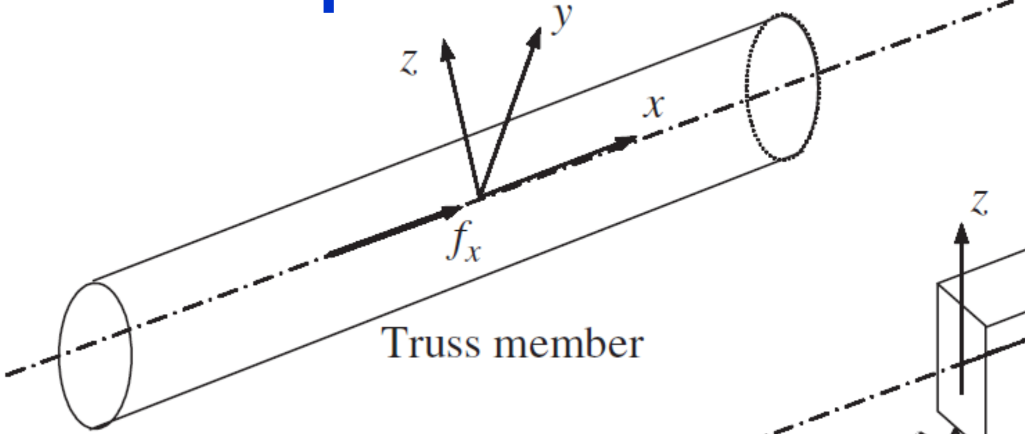
- Modeling of the geometry.
- Meshing (discretization).
- Specification of material property.
- Specification of boundary, initial and loading conditions.

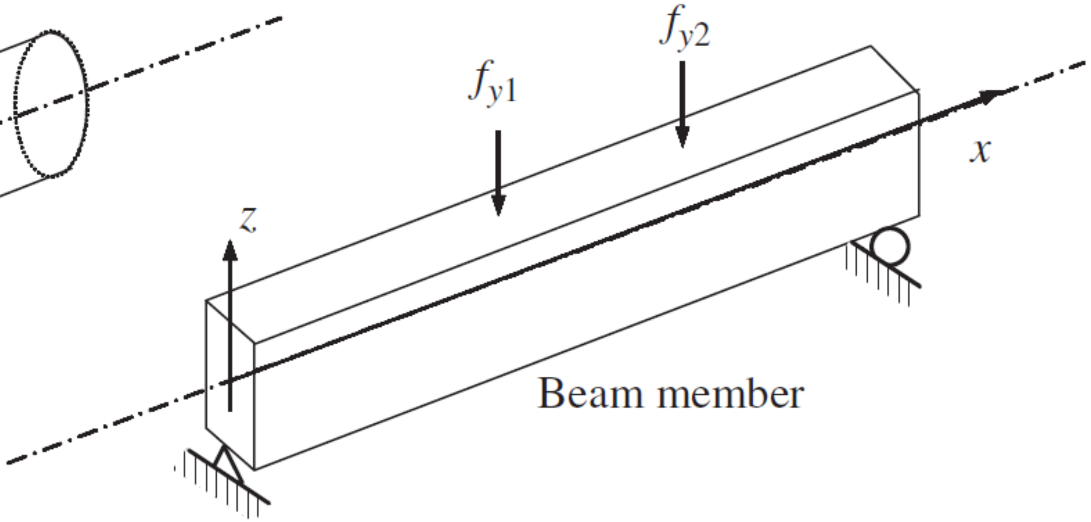


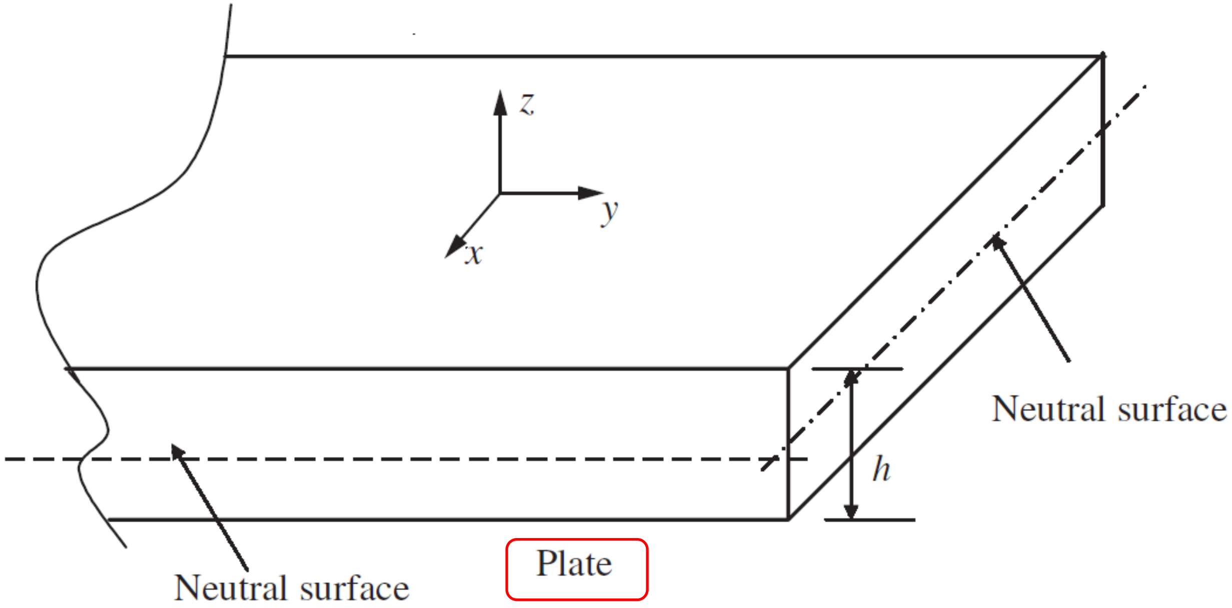
Structural components

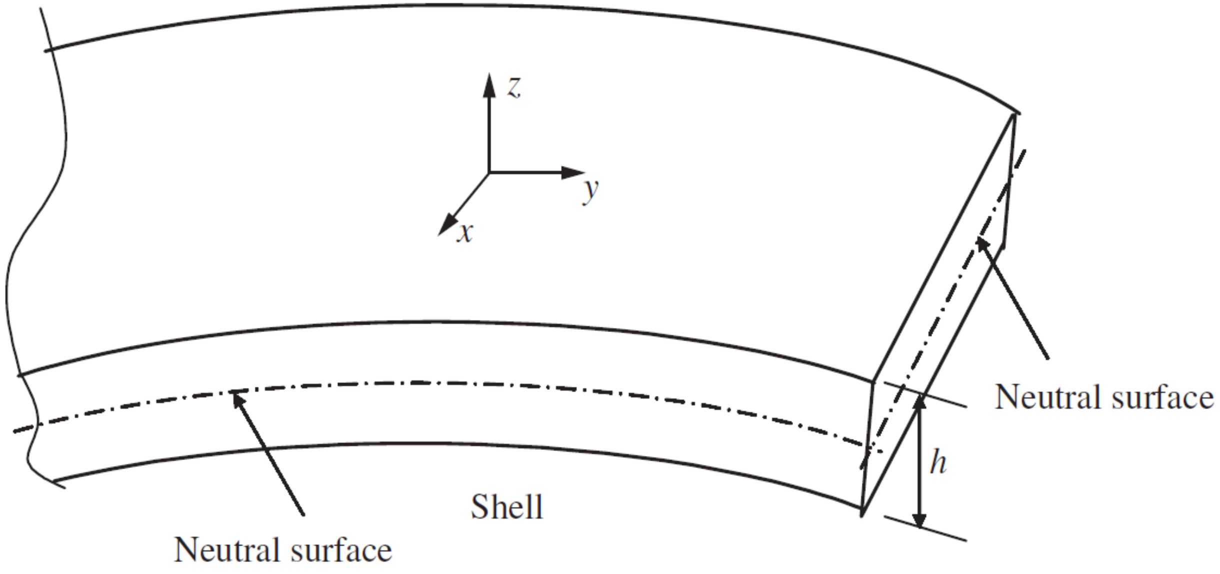
Structures are made of structural components that are in turn made of solids. There are generally four most commonly used structural components: truss, beam, plate, and shell. For the design of wind turbines, as in every engineering structure, the main purpose of using these structural components is to effectively utilize the material and reduce the weight and cost of the structure.

Structural components









Basic Definition: Deformation

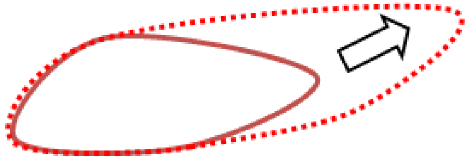
A Deformable object is defined by:

a) Undeformed shape (equilibrium configuration/rest shape/initial shape)

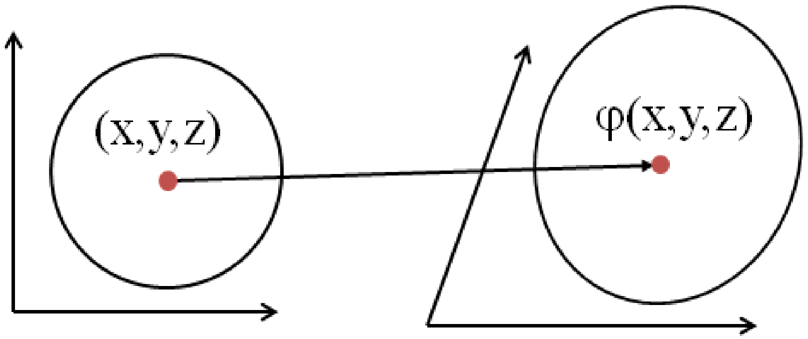
b) Set of material parameters that define how it deforms under applied forces

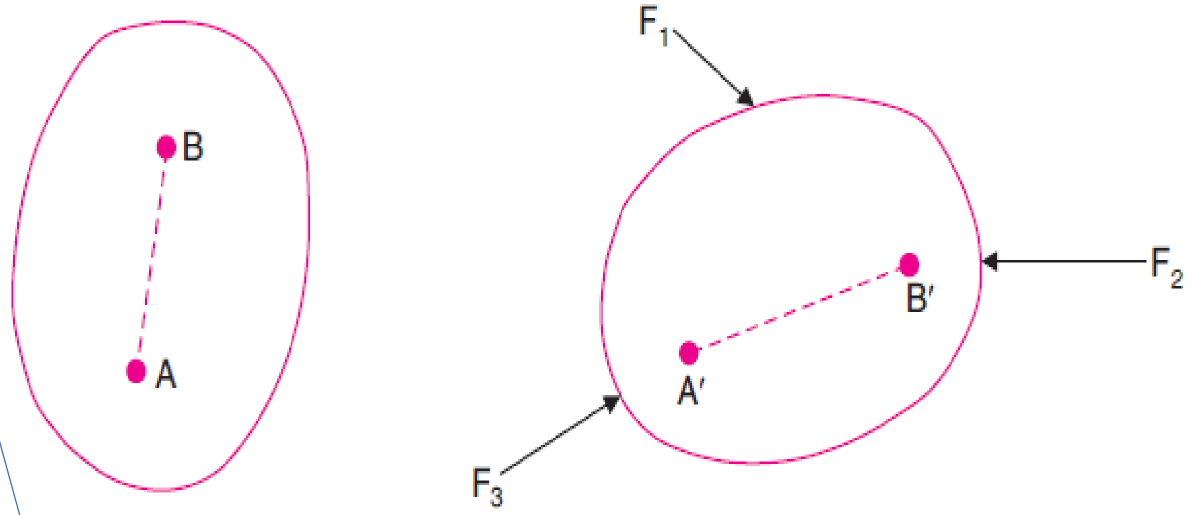
Deformation: A mapping of the positions of every particle in the original object to those in the deformed body

Modify Geometry



Space Transformation

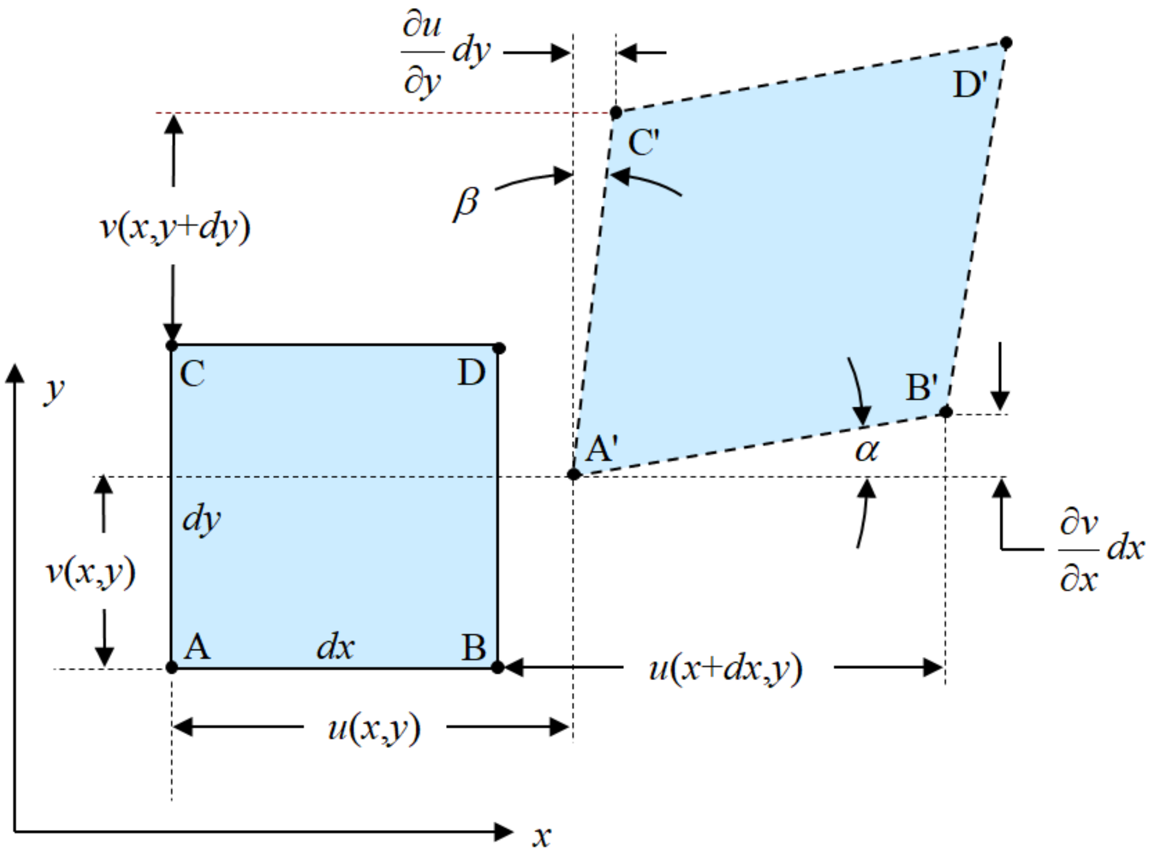




Two-Dimensional Theory

Deformation and Strain

Strain-Displacement Relations



$$e_x = \frac{\partial u}{\partial x}$$

$$e_y = \frac{\partial v}{\partial y}$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\gamma_{xy}$$

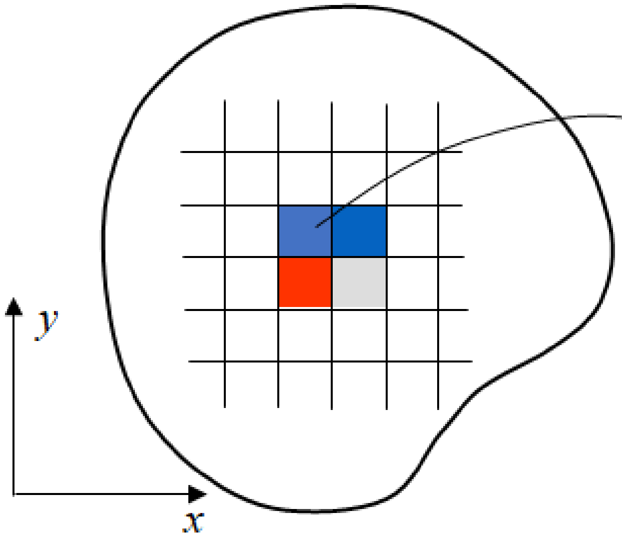
Three-Dimensional Theory ⇨

$$\mathbf{e} = [\mathbf{e}] = \begin{bmatrix} e_x & e_{xy} & e_{xz} \\ e_{yx} & e_y & e_{yz} \\ e_{zx} & e_{zy} & e_z \end{bmatrix}$$

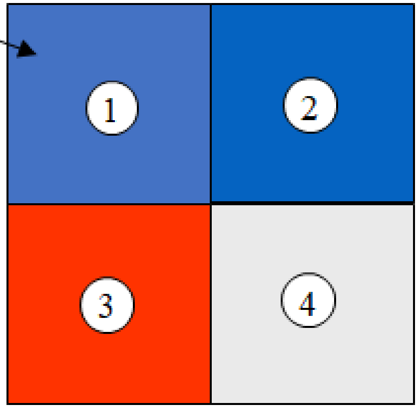
Strain Compatibility

Compatibility Equation

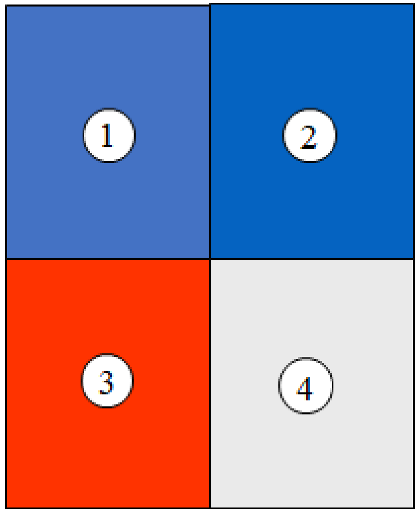
$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y}$$



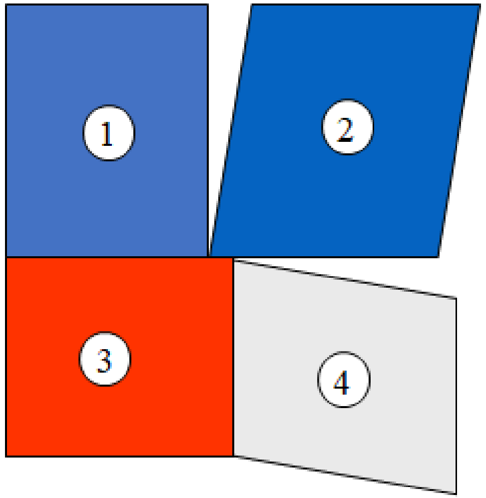
Discretized Elastic Solid



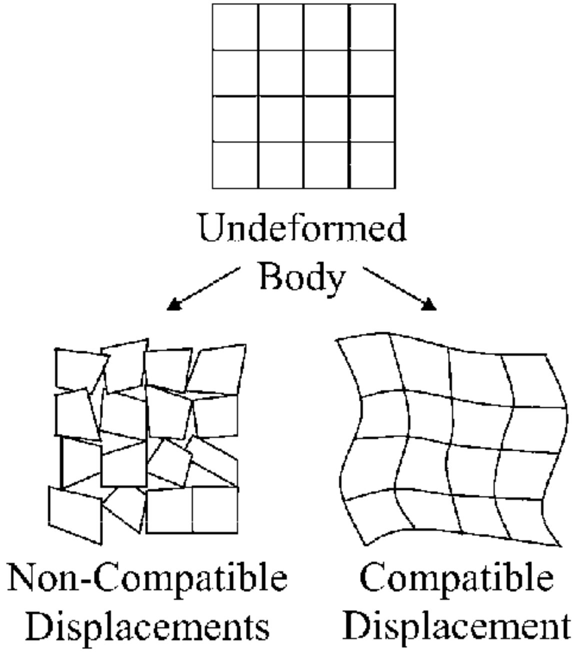
Undeformed Configuration



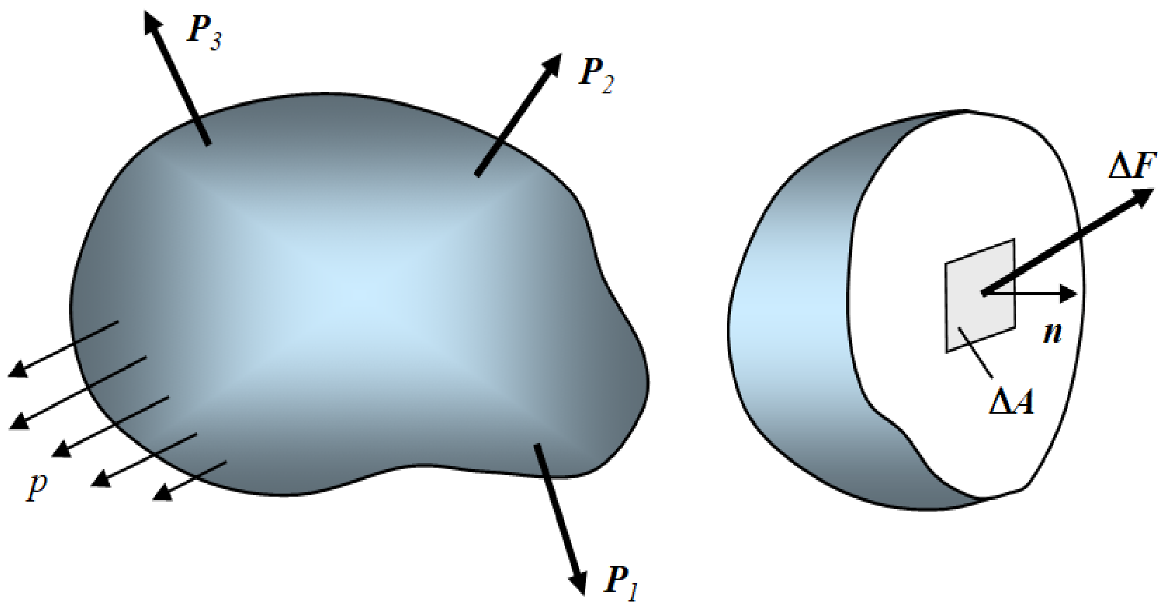
Deformed Configuration
Continuous Displacements



Deformed Configuration
Discontinuous Displacements



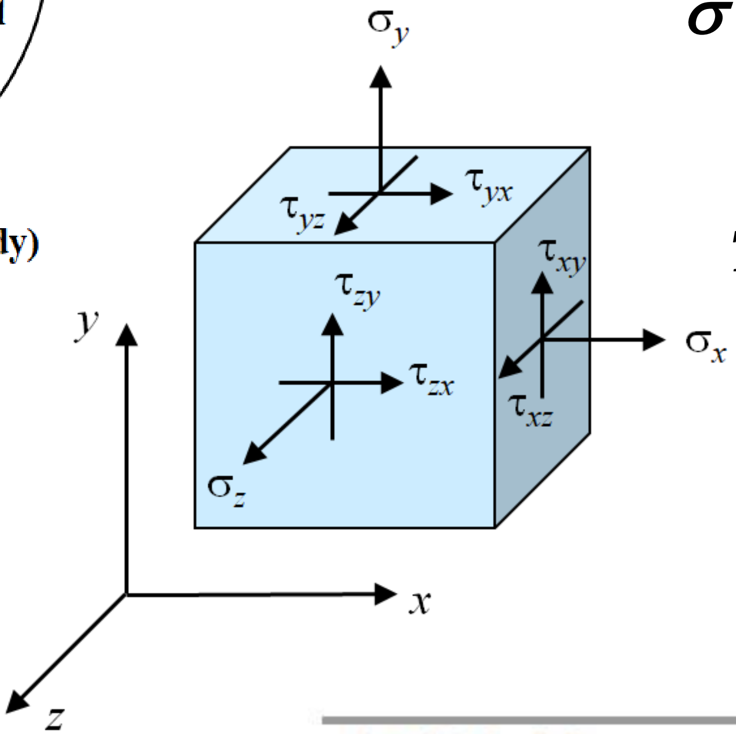
Traction and Stress



(Externally Loaded Body) (Sectioned Body)

Traction Vector

$$T^n(x,n) = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$



$$\boldsymbol{T}^{n}(\boldsymbol{x},\boldsymbol{n}=\boldsymbol{e_1}) = \sigma_x \boldsymbol{e_1} + \tau_{xy}\boldsymbol{e_2} + \tau_{xz}\boldsymbol{e_3}$$

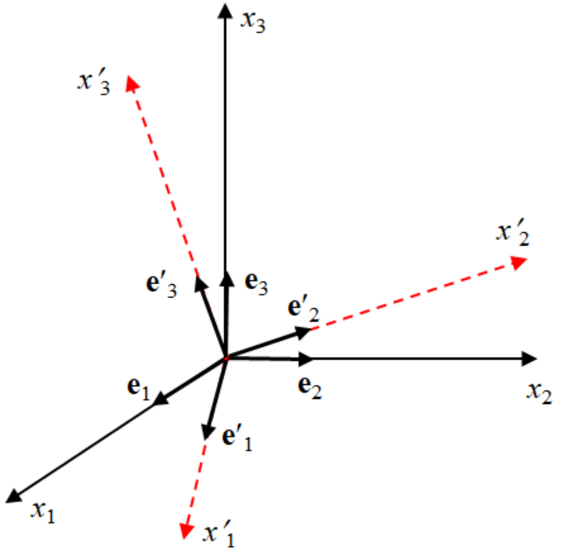
$$\boldsymbol{T}^{n}(\boldsymbol{x},\boldsymbol{n}=\boldsymbol{e_2}) = \tau_{yx} \boldsymbol{e_1} + \sigma_{y}\boldsymbol{e_2} + \tau_{yz}\boldsymbol{e_3}$$

$$\boldsymbol{T}^{n}(\boldsymbol{x},\boldsymbol{n}=\boldsymbol{e_3}) = \tau_{zx} \boldsymbol{e_1} + \tau_{zy}\boldsymbol{e_2} + \sigma_{z}\boldsymbol{e_3}$$

$$\boldsymbol{\sigma} = [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{T}^{n} &= (\sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z)\boldsymbol{e_1} \\ &+ (\tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z)\boldsymbol{e_2} \\ &+ (\tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z)\boldsymbol{e_3} \end{aligned}$$

Stress Transformation



$$\cos(x_i', x_j) = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

Three-Dimensional Transformation

$$\sigma_x' = \sigma_x l_1^2 + \sigma_y m_1^2 + \sigma_z n_1^2 + 2(\tau_{xy} l_1 m_1 + \tau_{yz} m_1 n_1 + \tau_{zx} n_1 l_1)$$

$$\sigma_y' = \sigma_x l_2^2 + \sigma_y m_2^2 + \sigma_z n_2^2 + 2(\tau_{xy} l_2 m_2 + \tau_{yz} m_2 n_2 + \tau_{zx} n_2 l_2)$$

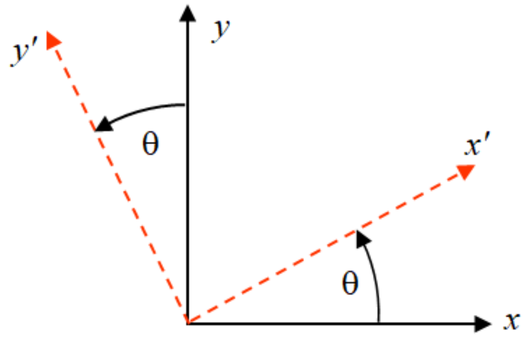
$$\sigma_z' = \sigma_x l_3^2 + \sigma_y m_3^2 + \sigma_z n_3^2 + 2(\tau_{xy} l_3 m_3 + \tau_{yz} m_3 n_3 + \tau_{zx} n_3 l_3)$$

$$\tau_{xy}' = \sigma_x l_1 l_2 + \sigma_y m_1 m_2 + \sigma_z n_1 n_2 + \tau_{xy}(l_1 m_2 + m_1 l_2) + \tau_{yz}(m_1 n_2 + n_1 m_2) + \tau_{zx}(n_1 l_2 + l_1 n_2)$$

$$\tau_{yz}' = \sigma_x l_2 l_3 + \sigma_y m_2 m_3 + \sigma_z n_2 n_3 + \tau_{xy}(l_2 m_3 + m_2 l_3) + \tau_{yz}(m_2 n_3 + n_2 m_3) + \tau_{zx}(n_2 l_3 + l_2 n_3)$$

$$\tau_{zx}' = \sigma_x l_3 l_1 + \sigma_y m_3 m_1 + \sigma_z n_3 n_1 + \tau_{xy}(l_3 m_1 + m_3 l_1) + \tau_{yz}(m_3 n_1 + n_3 m_1) + \tau_{zx}(n_3 l_1 + l_3 n_1)$$

Two-Dimensional Transformation



$$\sigma_x' = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta$$

$$\sigma_y' = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta\cos\theta$$

$$\tau_{xy}' = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

Principal Stresses and Directions

$$
\begin{aligned}
(\sigma_x-\lambda)n_1+\tau_{xy}n_2+\tau_{xz}n_2&=0\\
\tau_{xy}n_1+(\sigma_y-\lambda)n_2+\tau_{yz}n_3&=0\\
\tau_{xz}n_1+\tau_{yz}n_2+(\sigma_z-\lambda)n_3&=0
\end{aligned}
\Rightarrow
\begin{bmatrix}(\sigma_x-\lambda) & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & (\sigma_y-\lambda) & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & (\sigma_z-\lambda)\end{bmatrix}
\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix}=0
$$

Homogeneous System of Algebraic Equations, Non - Trival Solution

$$
\begin{vmatrix}(\sigma_x-\lambda) & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & (\sigma_y-\lambda) & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & (\sigma_z-\lambda)\end{vmatrix}=0 \Rightarrow -\lambda^3+I_1\lambda^2-I_2\lambda+I_3=0
$$

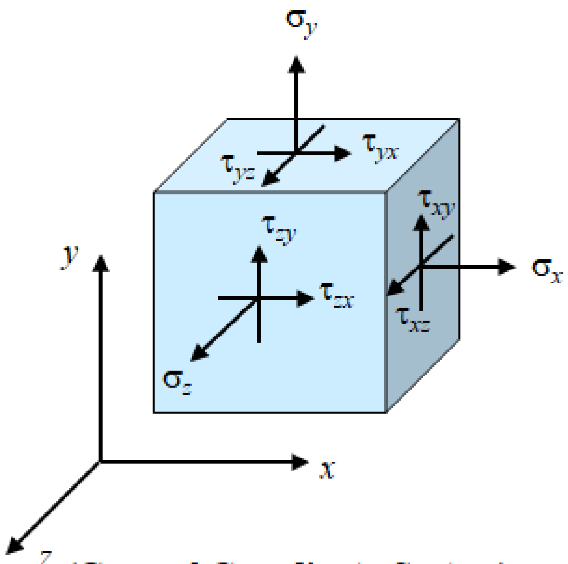
I_i = Fundamental Invariants

Roots of the characteristic equation are the principal stresses s_1 s_2 s_3
Corresponding to each principal stress is a principal direction $\boldsymbol{n}_1$ $\boldsymbol{n}_2$ $\boldsymbol{n}_3$
that can be used to construct a principal coordinate system

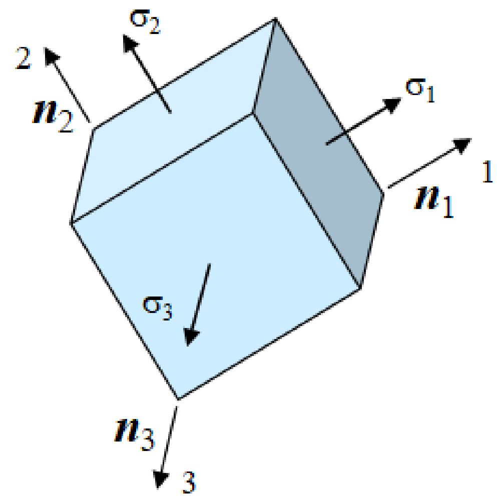
$$I_1=\sigma_{11}+\sigma_{22}+\sigma_{33}$$

$$I_2=\begin{vmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{11}\end{vmatrix}-\begin{vmatrix}\sigma_{11} & \sigma_{13}\\ \sigma_{13} & \sigma_{33}\end{vmatrix}-\begin{vmatrix}\sigma_{22} & \sigma_{23}\\ \sigma_{23} & \sigma_{33}\end{vmatrix}$$

$$I_3=\begin{vmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{12} & \sigma_{22} & \sigma_{23}\\ \sigma_{13} & \sigma_{23} & \sigma_{33}\end{vmatrix}$$



(General Coordinate System)



(Principal Coordinate System)

Hooke's Law

$$\sigma_x = C_{11}e_x + C_{12}e_y + C_{13}e_z + 2C_{14}e_{xy} + 2C_{15}e_{yz} + 2C_{16}e_{zx}$$
$$\sigma_y = C_{21}e_x + C_{22}e_y + C_{23}e_z + 2C_{24}e_{xy} + 2C_{25}e_{yz} + 2C_{26}e_{zx}$$
$$\sigma_z = C_{31}e_x + C_{32}e_y + C_{33}e_z + 2C_{34}e_{xy} + 2C_{35}e_{yz} + 2C_{36}e_{zx}$$
$$\tau_{xy} = C_{41}e_x + C_{42}e_y + C_{43}e_z + 2C_{44}e_{xy} + 2C_{45}e_{yz} + 2C_{46}e_{zx}$$
$$\tau_{yz} = C_{51}e_x + C_{52}e_y + C_{53}e_z + 2C_{54}e_{xy} + 2C_{55}e_{yz} + 2C_{56}e_{zx}$$
$$\tau_{zx} = C_{61}e_x + C_{62}e_y + C_{63}e_z + 2C_{64}e_{xy} + 2C_{65}e_{yz} + 2C_{66}e_{zx}$$

λ = *Lamé's constant*
μ = *shear modulus* or *modulus of rigidity*

E = *modulus of elasticity* or *Young's modulus*
ν = *Poisson's ratio*

Isotropic Homogeneous Materials

$$\sigma_x = \lambda(e_x + e_y + e_z) + 2\mu e_x$$
$$\sigma_y = \lambda(e_x + e_y + e_z) + 2\mu e_y$$
$$\sigma_z = \lambda(e_x + e_y + e_z) + 2\mu e_z$$
$$\tau_{xy} = 2\mu e_{xy}$$
$$\tau_{yz} = 2\mu e_{yz}$$
$$\tau_{zx} = 2\mu e_{zx}$$

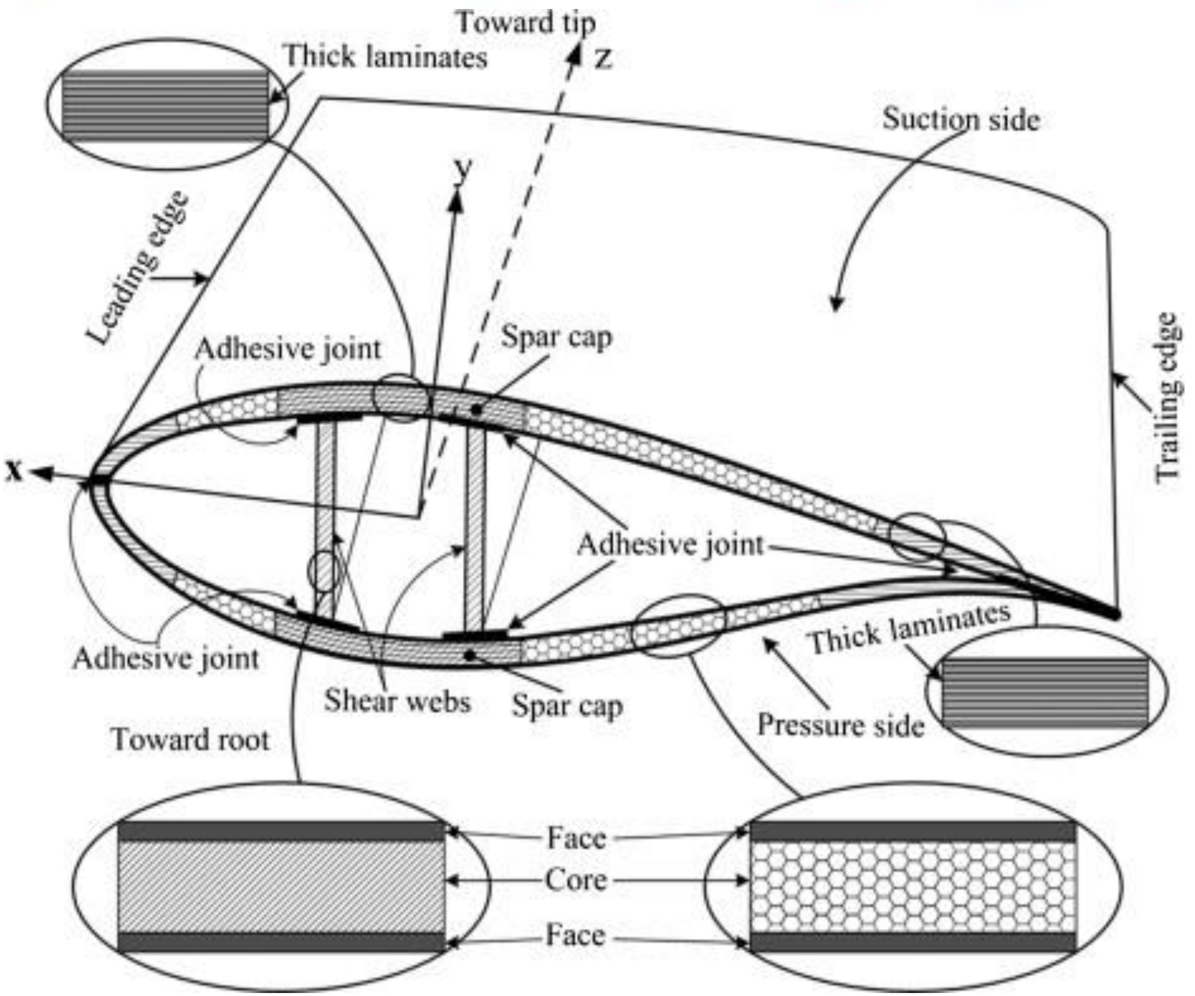
Wind turbine towers made of isotropic homogeneous material (steel)

$$e_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$
$$e_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right]$$
$$e_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$
$$e_{xy} = \frac{1+\nu}{E}\tau_{xy} = \frac{1}{2\mu}\tau_{xy}$$
$$e_{yz} = \frac{1+\nu}{E}\tau_{yz} = \frac{1}{2\mu}\tau_{yz}$$
$$e_{zx} = \frac{1+\nu}{E}\tau_{zx} = \frac{1}{2\mu}\tau_{zx}$$

Orthotropic Materials
(Three Planes of Material Symmetry)

$$
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}
=
\begin{bmatrix}
\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\
-\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\
-\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\
. & . & . & \frac{1}{\mu_{23}} & 0 & 0 \\
. & . & . & . & \frac{1}{\mu_{31}} & 0 \\
. & . & . & . & . & \frac{1}{\mu_{12}}
\end{bmatrix}
\begin{bmatrix} e_x \\ e_y \\ e_z \\ 2e_{yz} \\ 2e_{zx} \\ 2e_{xy} \end{bmatrix}
$$

Nine Independent Elastic Constants for 3-D
Four Independent Elastic Constants for 2-D



Wind turbine blades made of composite materials (laminates) with orthotropic behaviour

Linear Elastic Constitutive Solid Model

Develop Force-Deformation Constitutive Equation in the Form of Stress-Strain Relations Under the Assumptions:

- Solid recovers original configuration when loads are removed
- Linear relation between stress and strain
- Neglect rate and history dependent behavior
- Include only mechanical loadings
- Thermal, electrical, pore-pressure, and other loadings can also be included as special cases

Nonlinear Behaviour

Sources of nonlinearity

- Nature is nonlinear and abounds with nonlinear systems.
- Nonlinear systems are those for which the principle of superposition does not hold.
- The sources of nonlinearities can be material (or constitutive), geometric, initial and/or boundary conditions.
- The nonlinearities may appear in:
  - the governing partial-differential equations,
  - the boundary conditions,
  - or both.

Sources of nonlinearity

Material nonlinearity / Boundary nonlinearity / Geometric nonlinearity

Material is one of the most common forms of nonlinearity, and would include nonlinear elastic (e.g. rubber), plastic (e.g., steel, concrete, soil), and viscoelastic or viscoplastic behavior (e.g. asphalt).

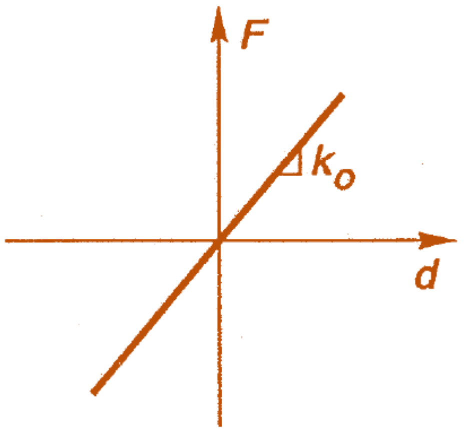
For thermal problems, a temperature dependent thermal conductivity will produce nonlinear equations.

Sources of nonlinearity

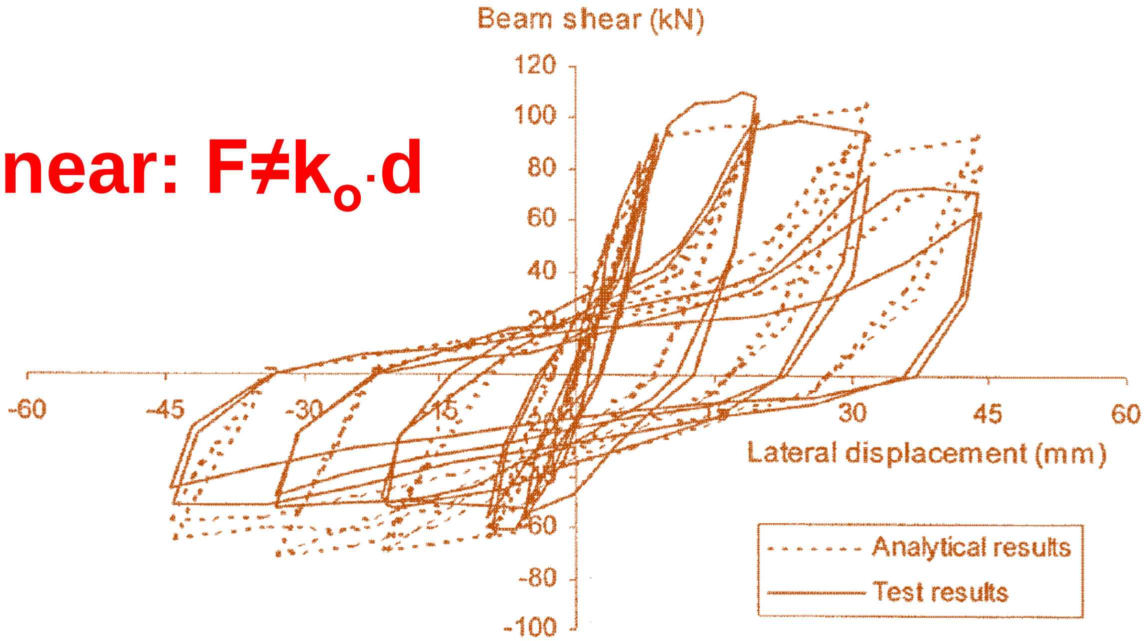
Material nonlinearity / Boundary nonlinearity / Geometric nonlinearity

The constitutive or material nonlinearity occurs when the stresses are nonlinear functions of the strains.

Nonlinear: $F \neq k_o \cdot d$



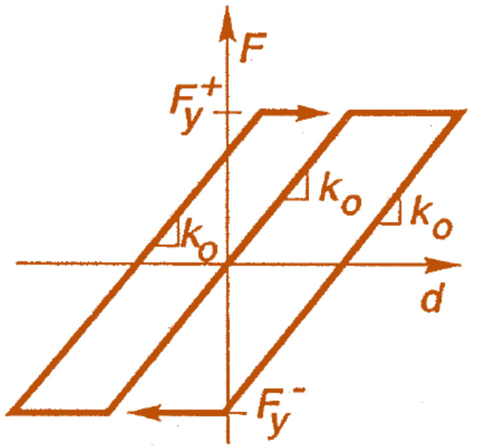
Linear: $F = k_o \cdot d$



Sources of nonlinearity

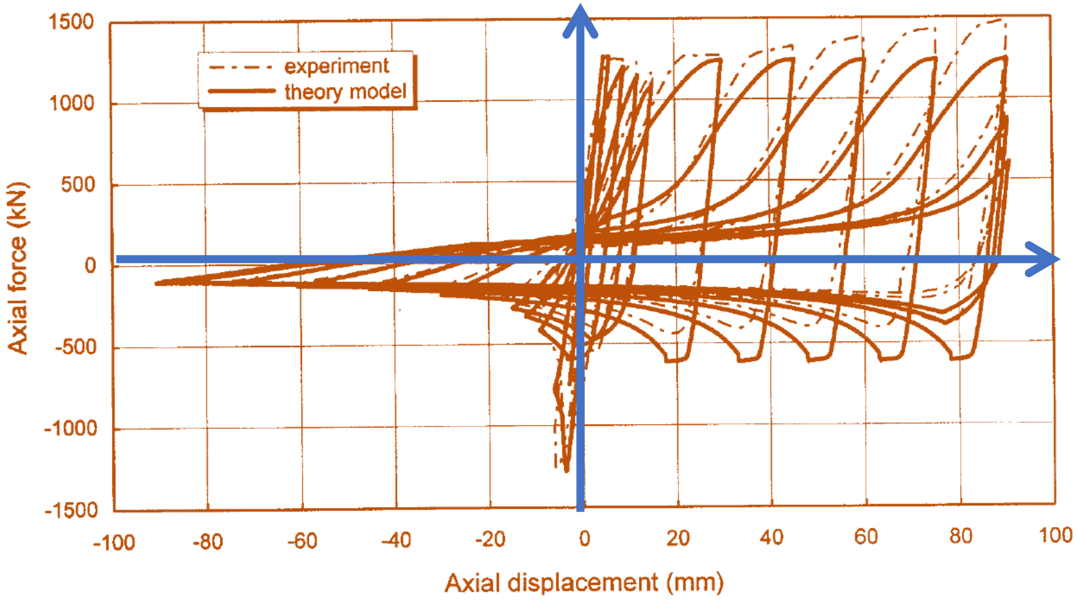
Material nonlinearity / Boundary nonlinearity / Geometric nonlinearity

Many materials (e.g., most metals) have a fairly linear stress/strain relationship at low strain values; but at higher strains the material yields, at which point the response becomes nonlinear and irreversible.



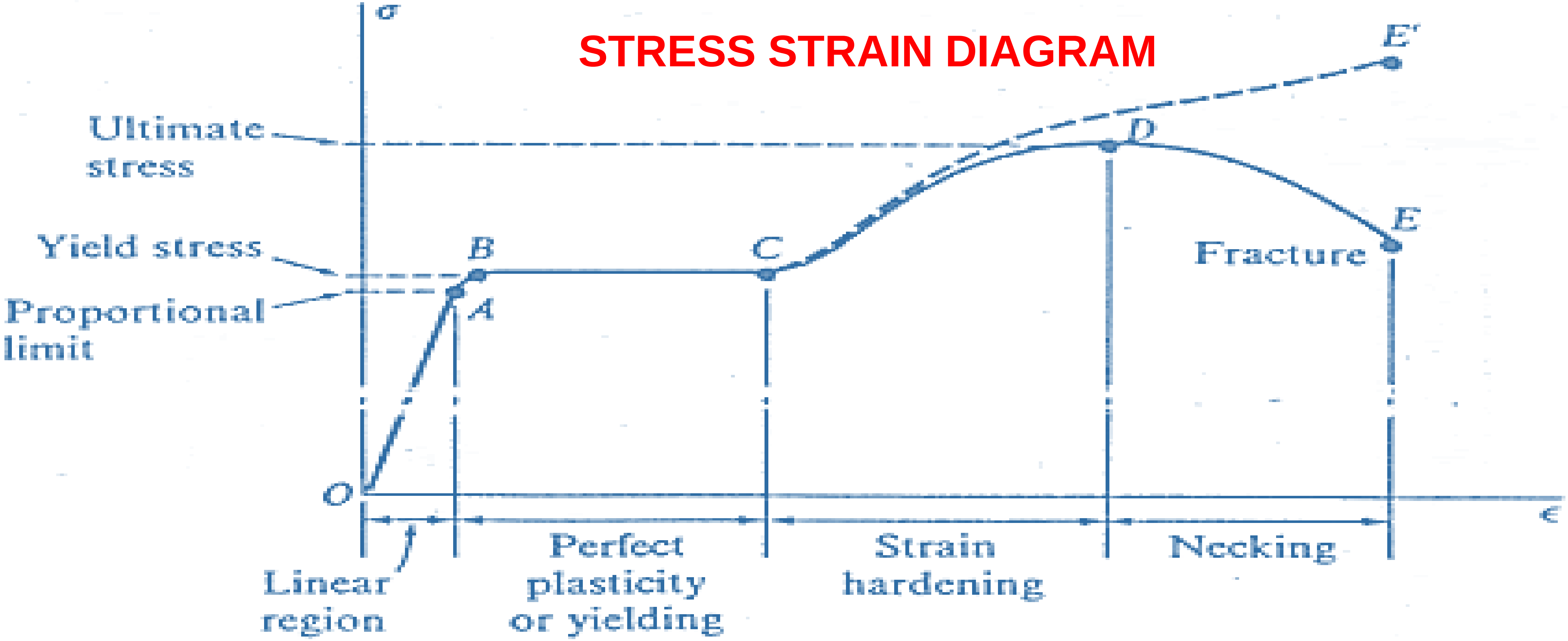
Elastoplastic behavior

Remennikov model for braces



Sources of nonlinearity

Material nonlinearity / Boundary nonlinearity / Geometric nonlinearity



Sources of nonlinearity

Material nonlinearity / **Boundary nonlinearity** / Geometric nonlinearity

Boundary nonlinearity is also known as Constraint and Contact Nonlinearity. Constraint nonlinearity in a system can occur if kinematic constraints are present in the model. The kinematic degrees-of-freedom of a model can be constrained by imposing restrictions on its movement.

Sources of nonlinearity

Material nonlinearity / **Boundary nonlinearity** / Geometric nonlinearity

For example, sand can sustain compressive loads but its tensile strength is zero. This is a typical (nonlinear) **contact problem** where the principle of superposition is not valid.

Sources of nonlinearity

Material nonlinearity / **Boundary nonlinearity** / Geometric nonlinearity

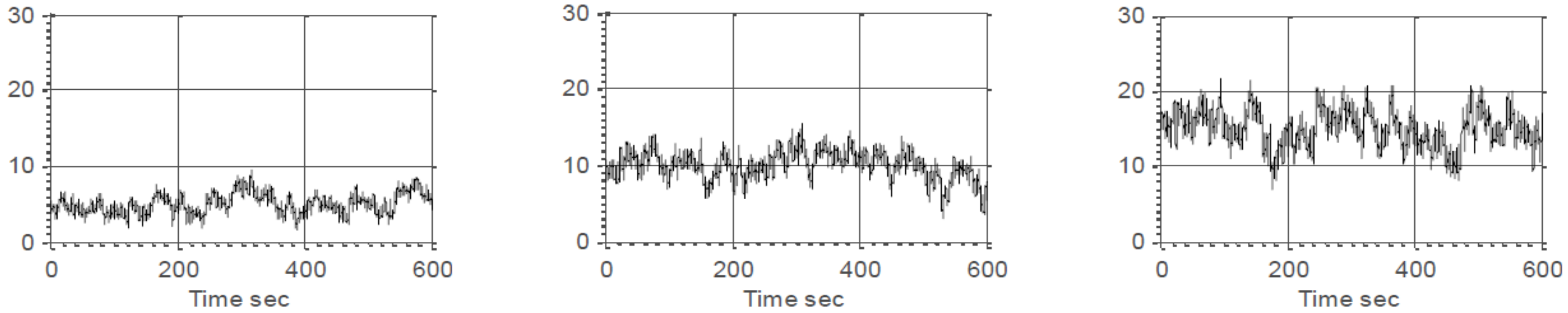
Contact: Include effects of contact is a type of "changing status" nonlinearity, where an abrupt change in stiffness may occur when bodies come into or out of contact with each other.

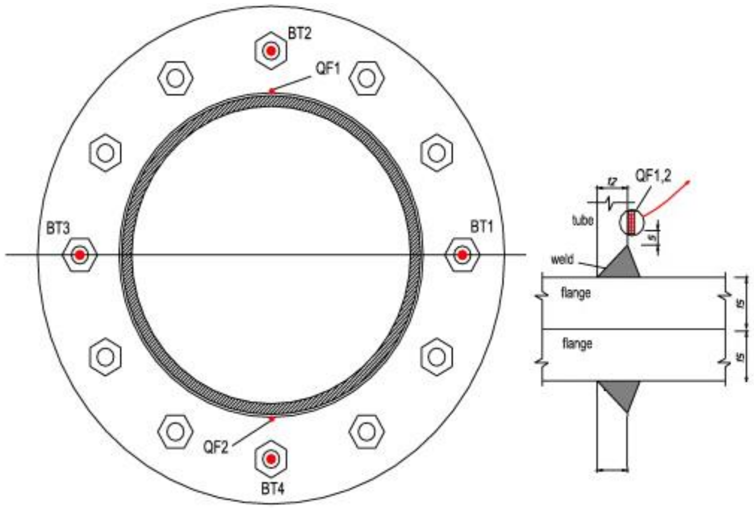
Sources of nonlinearity

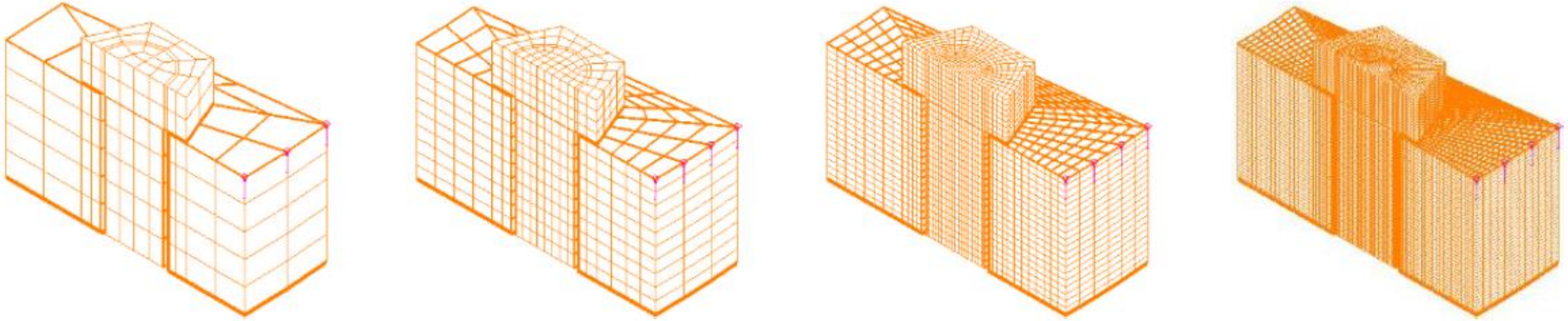
Material nonlinearity / **Boundary nonlinearity** / Geometric nonlinearity

Application: wind turbine tower double ring flanges that are bolted together with preloaded bolts

10-minutes wind velocity (fatigue load)





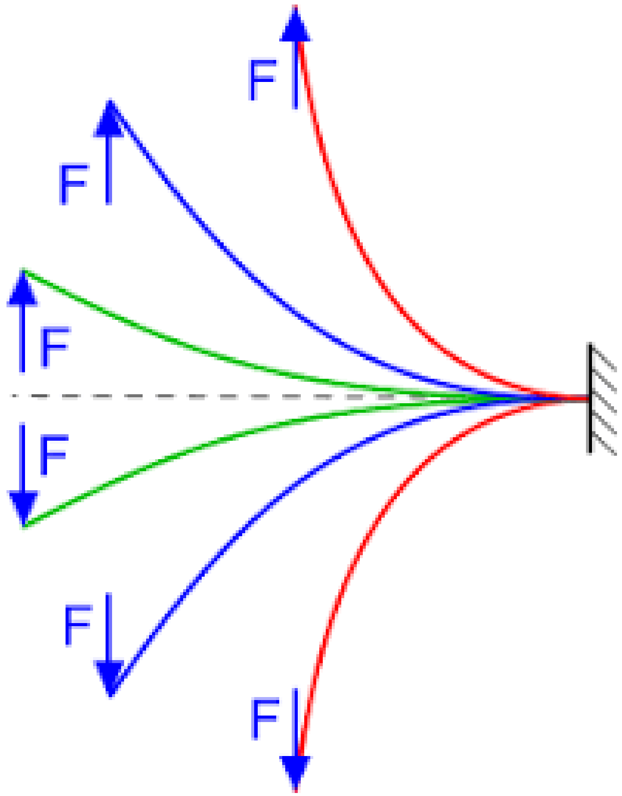


Finite element simulation models for bolt and flange interaction

Sources of nonlinearity

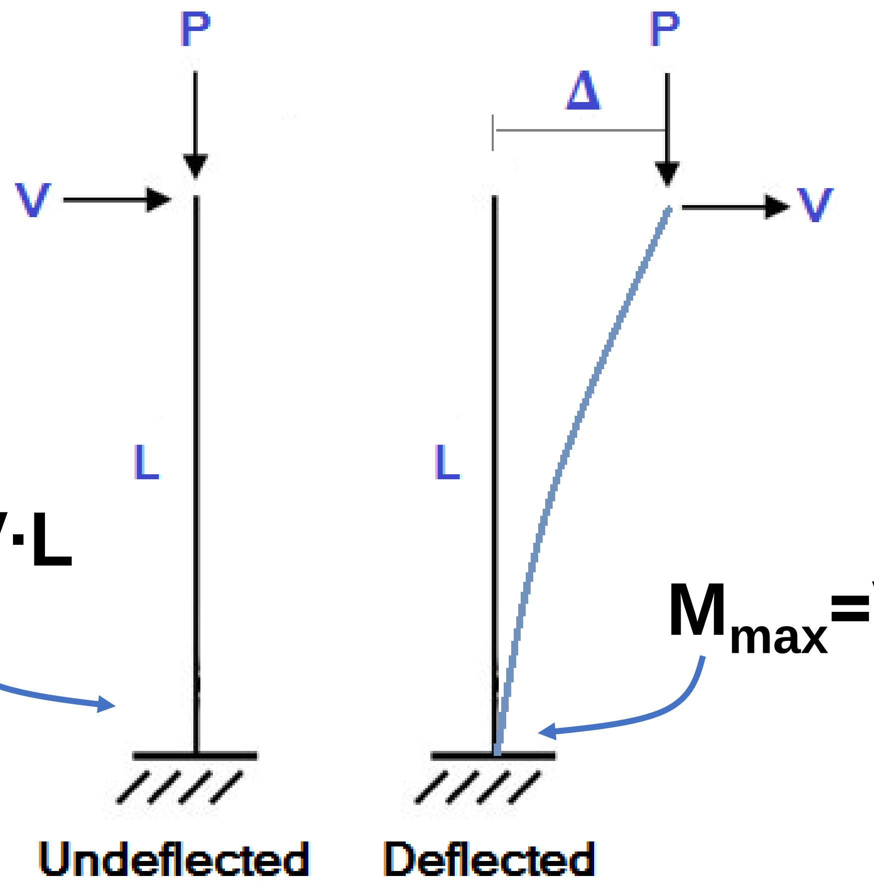
Material nonlinearity / Boundary nonlinearity / **Geometric nonlinearity**

The geometric nonlinearity is associated with large deformations in solids, resulting in nonlinear strain-displacement relations (e.g., mid-plane stretching, large curvatures of structural elements, large strains, and large rotations of elements). For example, consider a cantilever beam loaded vertically at the tip.



Sources of nonlinearity

Material nonlinearity / Boundary nonlinearity / **Geometric nonlinearity**



$$M_{max}=V\cdot L$$

$$M_{max}=V\cdot L+P\cdot\Delta$$

<u>Large deformations</u>

(Displacements are ‘large’,

although that strains are ‘small’.

Sources of nonlinearity

Material nonlinearity / Boundary nonlinearity / **Geometric nonlinearity**

One of the most important issue of geometric nonlinearity is the ***Instability***

Instability can be defined as the change in geometry of a structure or structural component under compression – resulting in loss of ability to resist loading.

- Every structure is in equilibrium – static or dynamic. If it is not in equilibrium, the body will be in motion or a mechanism.
- A mechanism cannot resist loads.
- Stability qualifies the state of equilibrium of a structure. Whether it is in stable or unstable equilibrium.

Sources of nonlinearity

Material nonlinearity / Boundary nonlinearity / **Geometric nonlinearity**

Instability

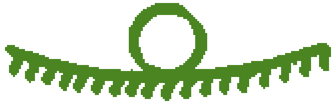
- Structure is in **stable equilibrium** when small perturbations do not cause large movements like a mechanism. Structure vibrates about it equilibrium position.
- Structure is in **unstable equilibrium** when small perturbations produce large movements – and the structure never returns to its original equilibrium position.
- Thus, stability talks about the equilibrium state of the structure.

Sources of nonlinearity

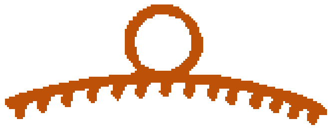
Material nonlinearity / Boundary nonlinearity / **Geometric nonlinearity**

Instability

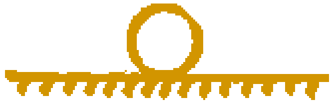
- Structure is in **stable equilibrium** when small perturbations do not cause large movements like a mechanism. Structure vibrates about it equilibrium position.
- Structure is in **unstable equilibrium** when small perturbations produce large movements – and the structure never returns to its original equilibrium position.
- Thus, stability talks about the equilibrium state of the structure.



Stable state



Unstable state



Neutral state

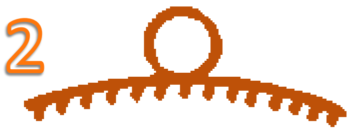
Sources of nonlinearity

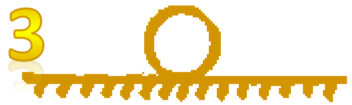
Material nonlinearity / Boundary nonlinearity / **Geometric nonlinearity**

Change in geometry of structure under compression – that results in its ability to resist loads – called instability. **Not true – this is called buckling.**

Buckling is a phenomenon that can occur for structures under compressive loads. The structure deforms and is in stable equilibrium in state-1. As the load increases, the structure suddenly changes to deformation state-2 at some critical load P_{cr}.

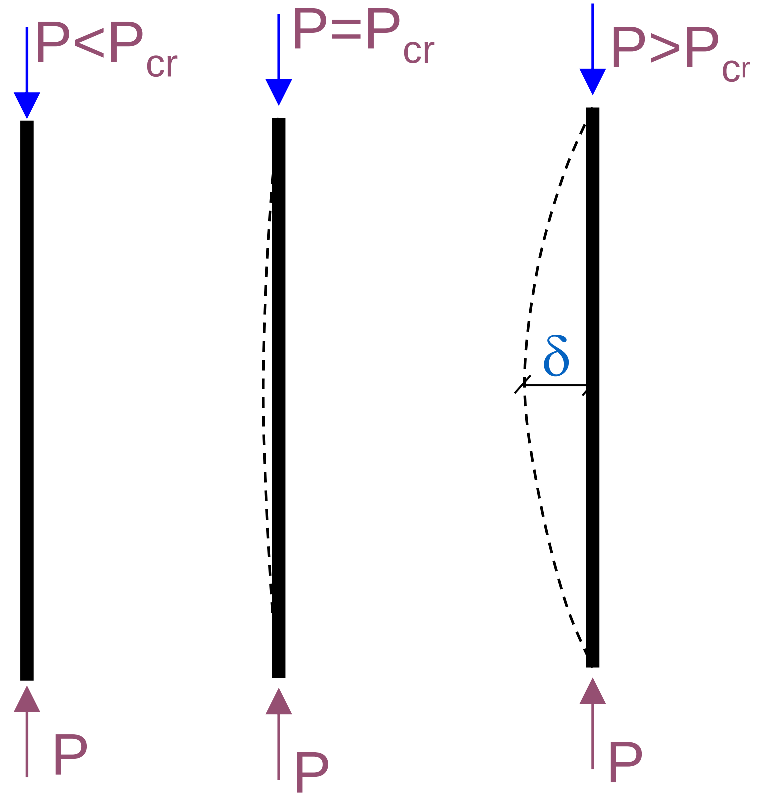
Buckling

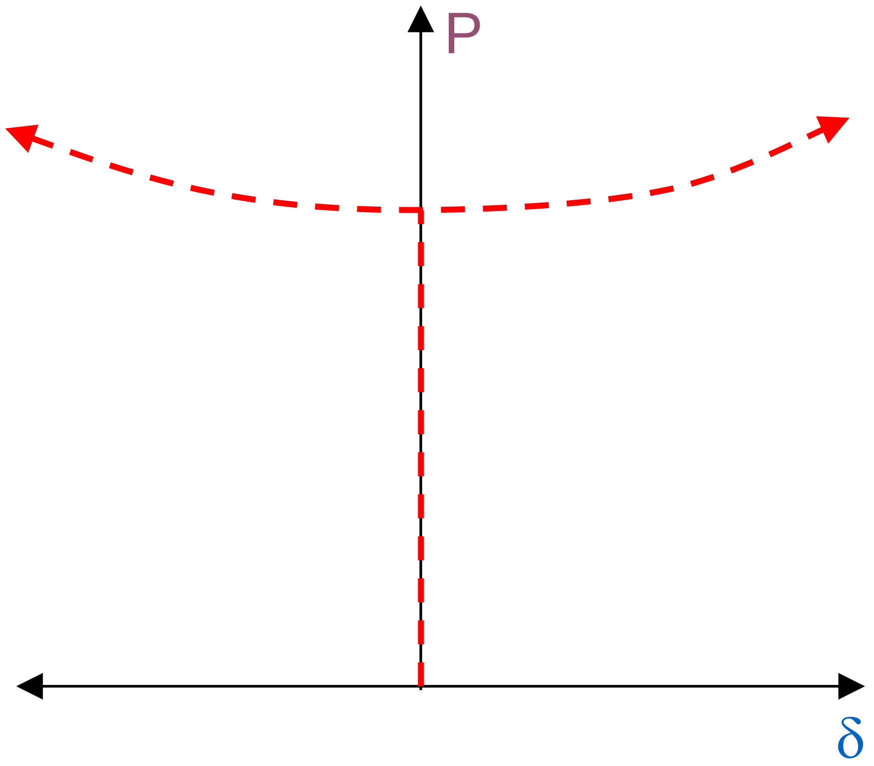


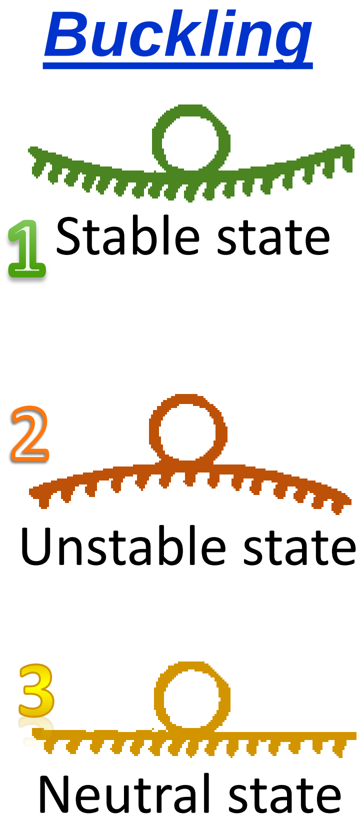


Sources of nonlinearity

Material nonlinearity / Boundary nonlinearity / **Geometric nonlinearity**



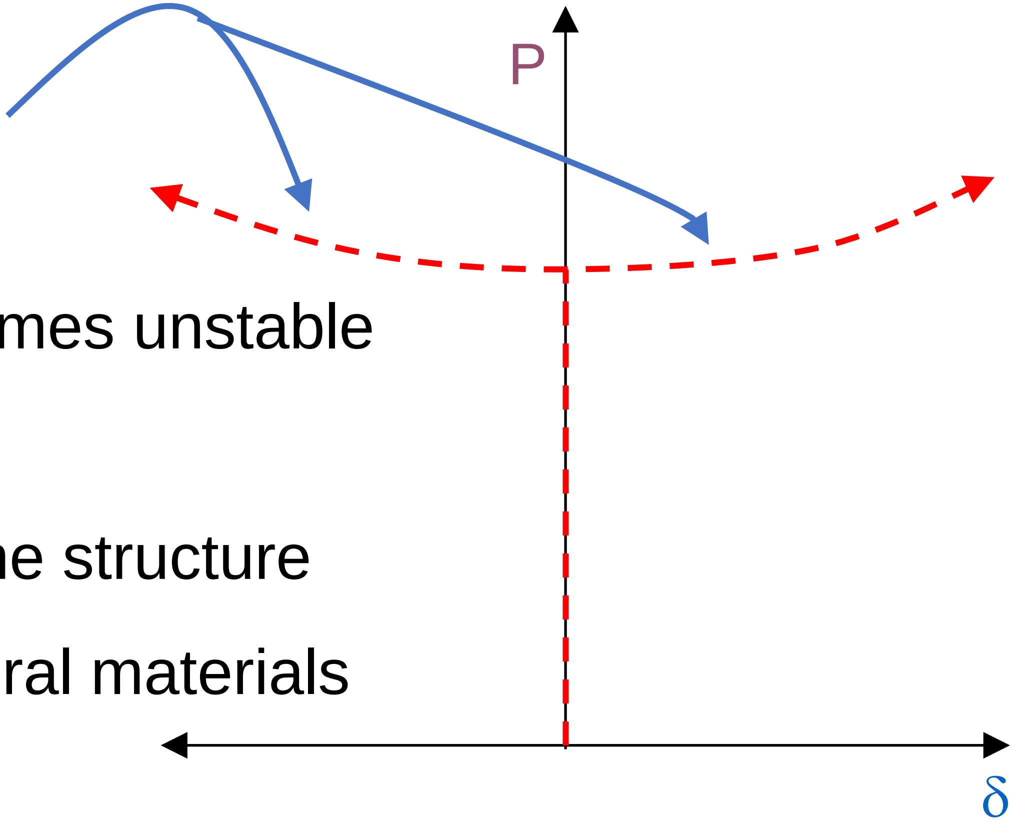




Sources of nonlinearity

Material nonlinearity / Boundary nonlinearity / **Geometric nonlinearity**

Bifurcation



The equilibrium state becomes unstable

due to:

- Large deformations of the structure
- Inelasticity of the structural materials

Buckling

Sources of nonlinearity

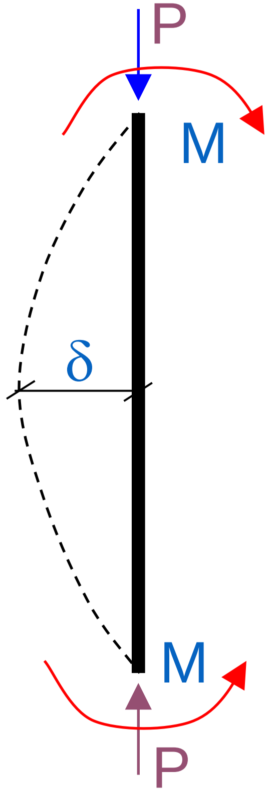
Material nonlinearity / Boundary nonlinearity / **Geometric nonlinearity**

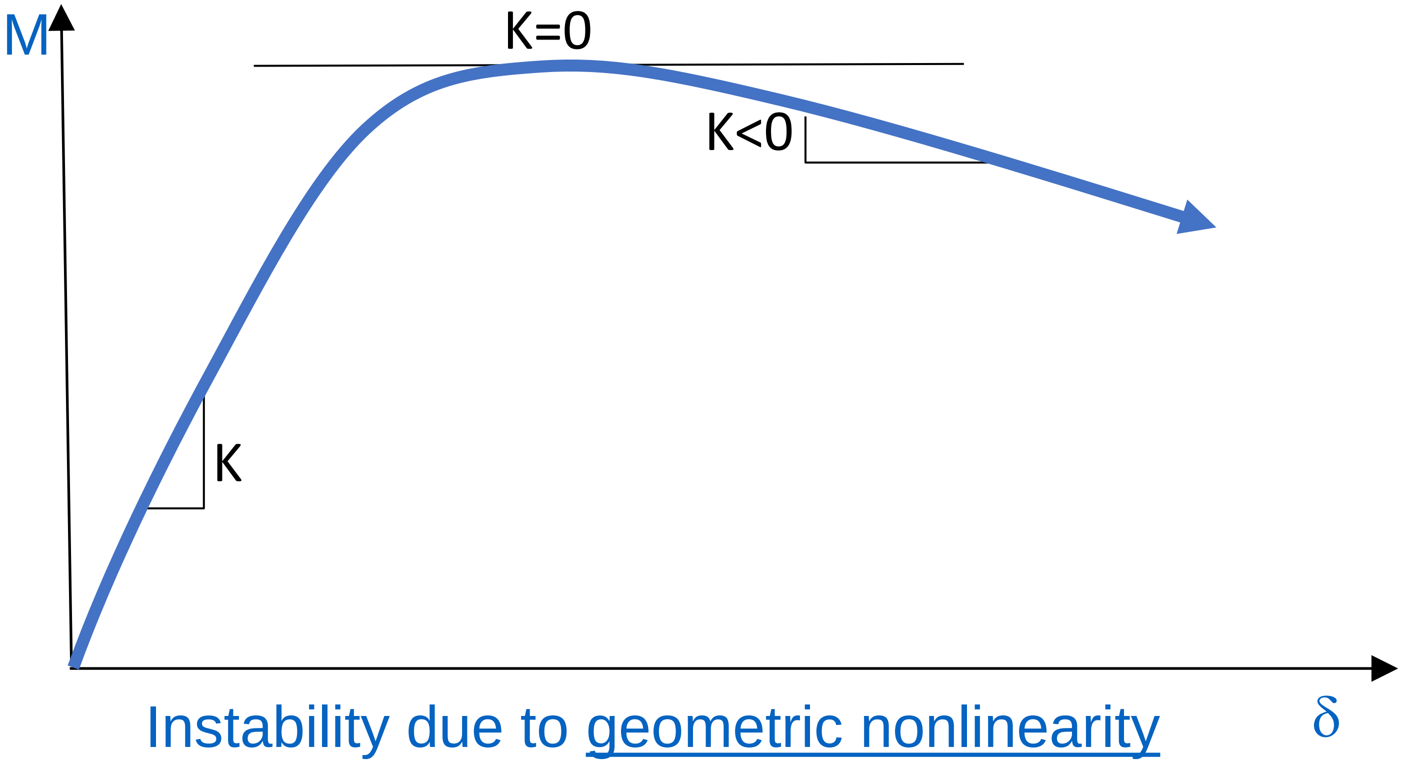
- The structure stiffness decreases as the loads are increased. The change is stiffness is due to large deformations.
- The structure stiffness decreases to zero and becomes negative.
- The load capacity is reached when the stiffness becomes zero.
- Neutral equilibrium when stiffness becomes zero and unstable equilibrium when stiffness is negative.
- Structural stability failure – when stiffness becomes negative.

Buckling

Sources of nonlinearity

Material nonlinearity / Boundary nonlinearity / **Geometric nonlinearity**





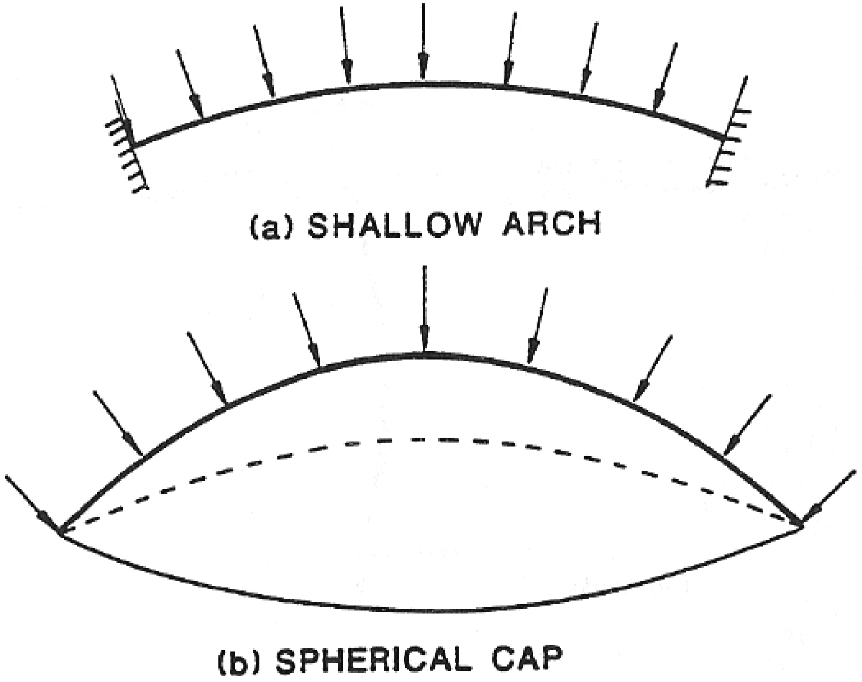
Instability due to geometric nonlinearity

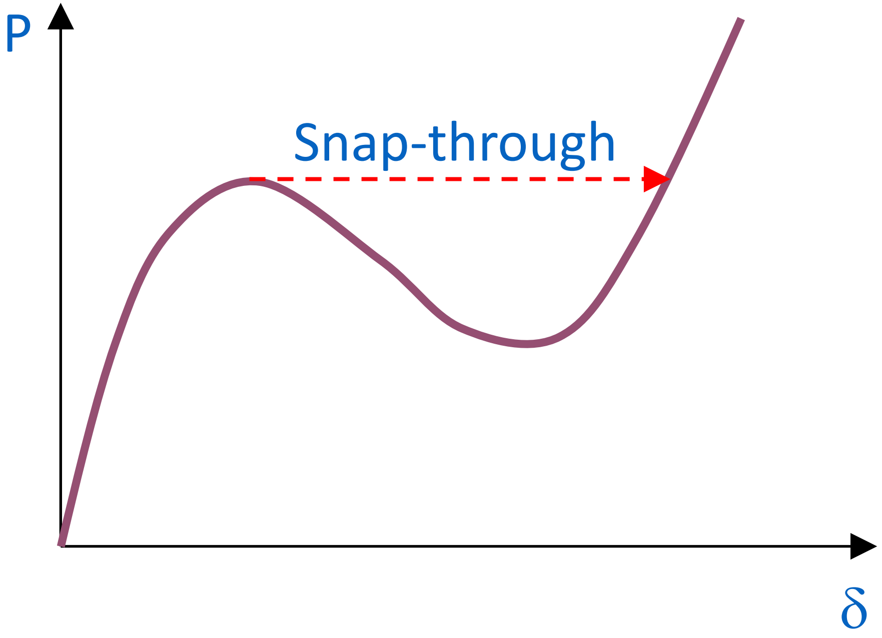
Buckling

Sources of nonlinearity

Material nonlinearity / Boundary nonlinearity / **Geometric nonlinearity**

Snap-through buckling



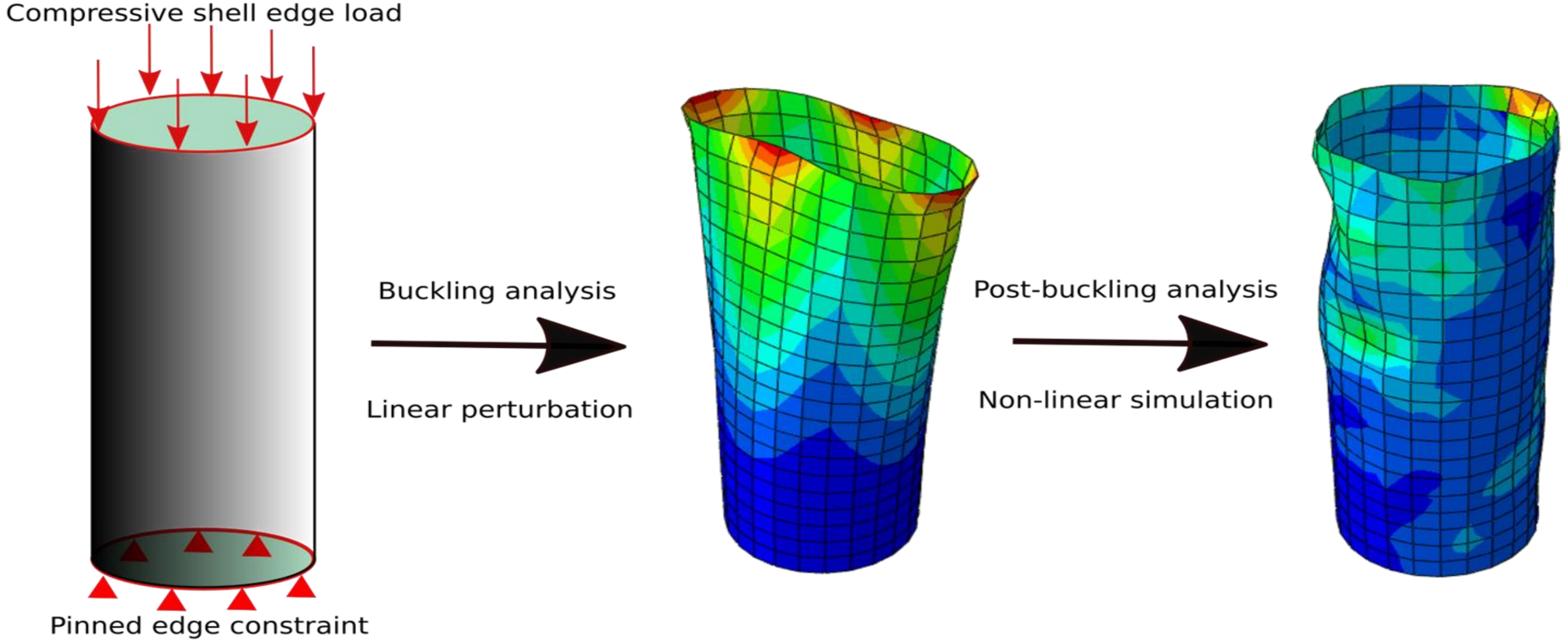


Buckling

Sources of nonlinearity

Material nonlinearity / Boundary nonlinearity / **Geometric nonlinearity**

Shell Buckling failure – very sensitive to imperfections



Buckling

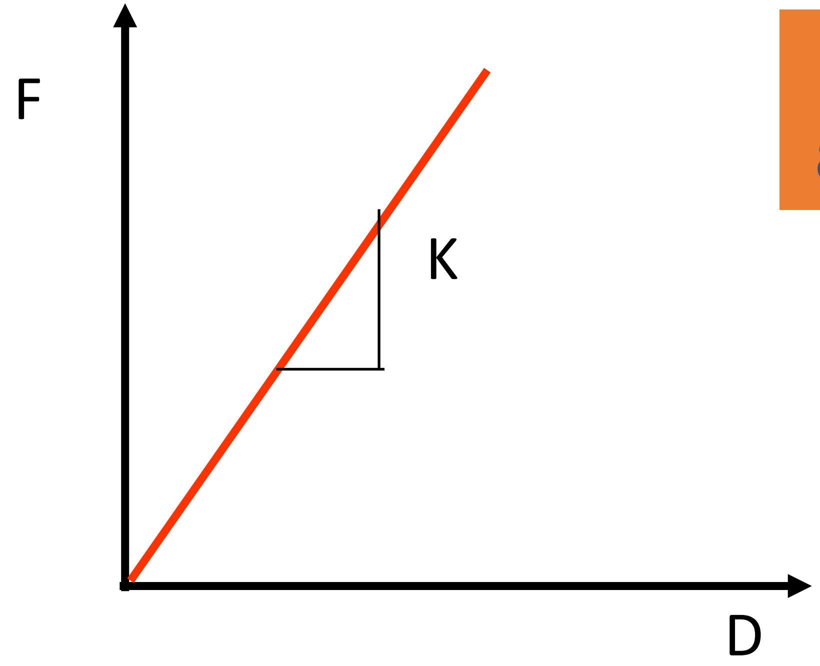
Solution of linear and nonlinear problems

In a **linear analysis**, the solution is calculated directly in one step by solving a system of linear equations. However, in order to trace the solution path in a **nonlinear analysis**, the load (or prescribed displacement) should be divided into a series of smaller increments. For each load increment equilibrium, a solution is found by performing several iterations, each of which is computationally comparable to a solution of a linear system. Therefore, nonlinear analysis may be computationally much more demanding compared to the linear analysis.

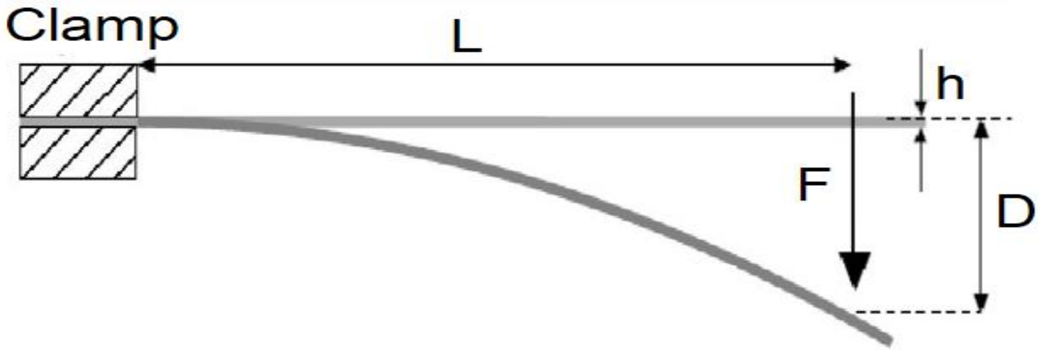
Linear analysis

Assuming linear static structural analysis, the matrix equation solved for is Hooke's Law:

$$[K]\{D\} = \{F\}$$



Because [K] is assumed to be constant, essentially only linear behavior is allowed As shown on the figure, if the force doubles, the displacement (and stresses) are assumed to double in linear analysis (Principle of Superposition). In many real-world situations, however, this small-displacement theory may not be valid. In these situations, nonlinear analysis may be required.



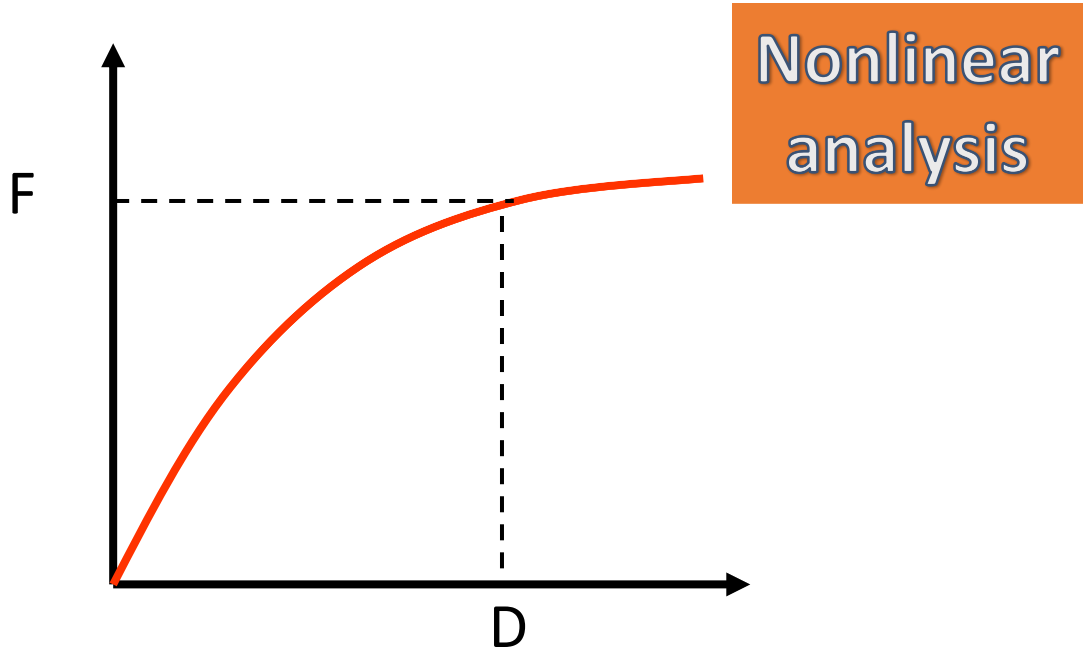
Background

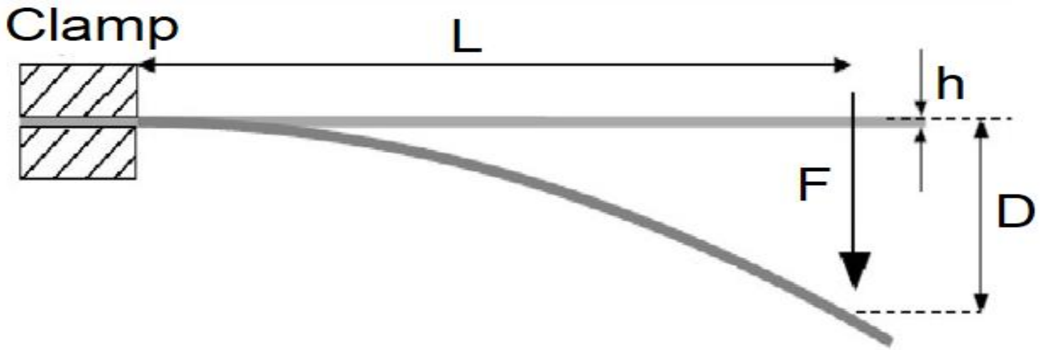
In a *nonlinear analysis*, the stiffness [K] is dependent on the displacement {x}:

$$[K(D)]\{D\} = \{F\}$$

The resulting force vs. displacement curve may be nonlinear, so doubling the force does not necessarily result in doubling of the displacements and stresses.

A nonlinear analysis is an iterative solution because this relationship between load (F) and response (x) is not known beforehand.





Linear Problem:

$$[K]\{D\}=\{\mathrm{P}\}$$
$$[K]\neq[K(\{D\})]$$
$$\{P\}\neq\{P(\{D\})\}$$

Nonlinear Problem:

$$[K]\{D\}=\{P\}$$
$$[K]=[K(\{D\})]$$
$$\{P\}=\{P(\{D\})\}$$

$$\frac{P}{u} = k = k_0 + k_N$$

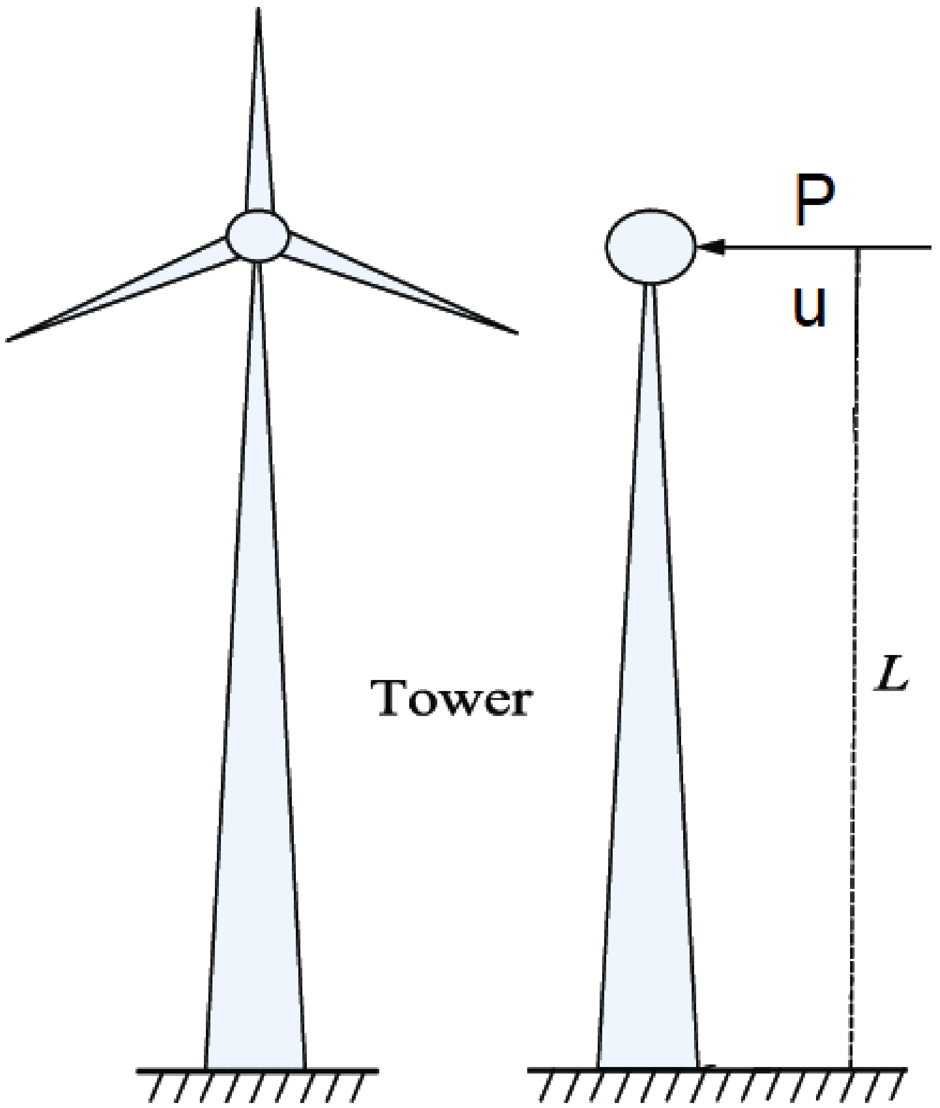
k_0 constant

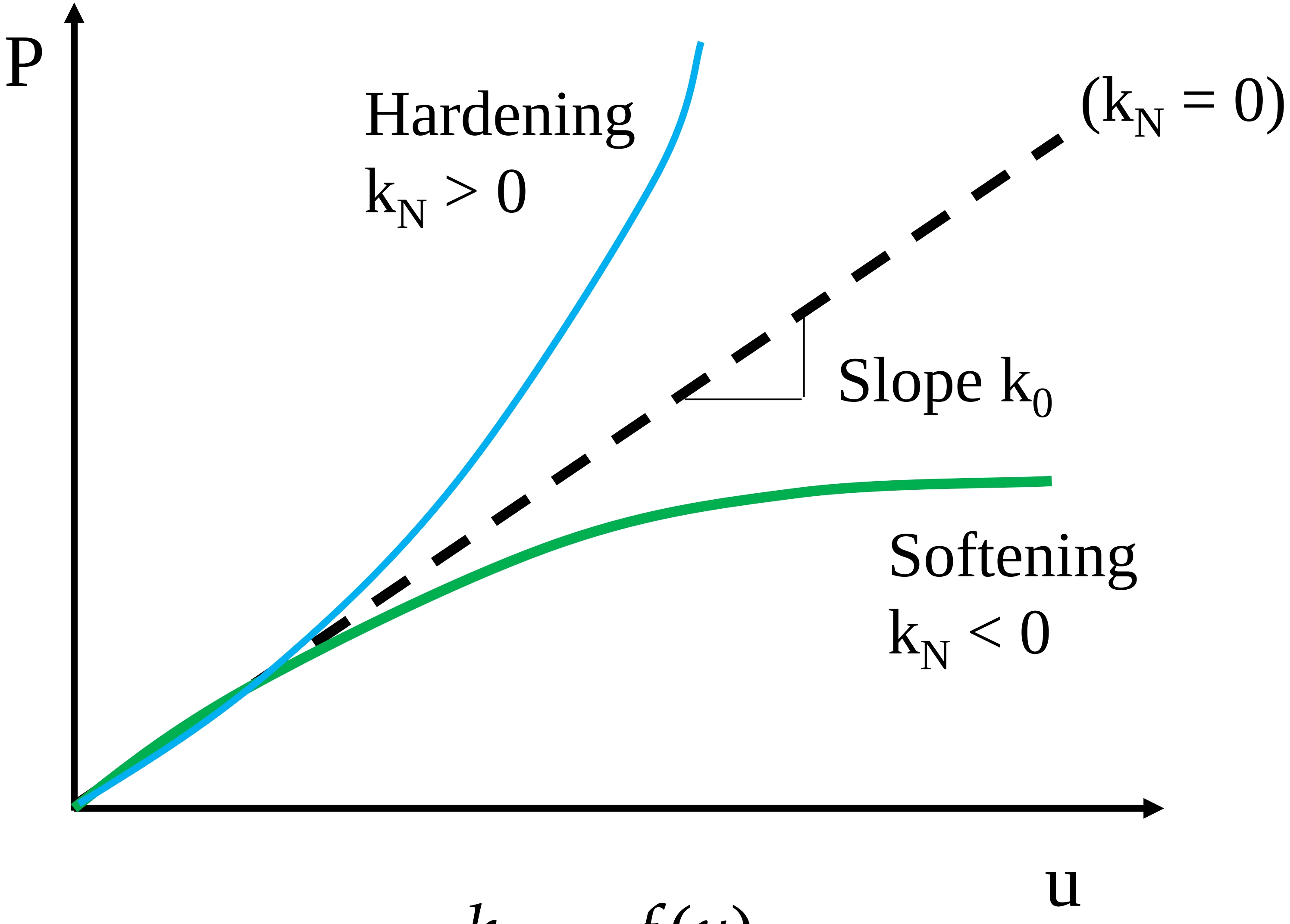
k_N function of u

Given P find u. Assume f(u) is known.

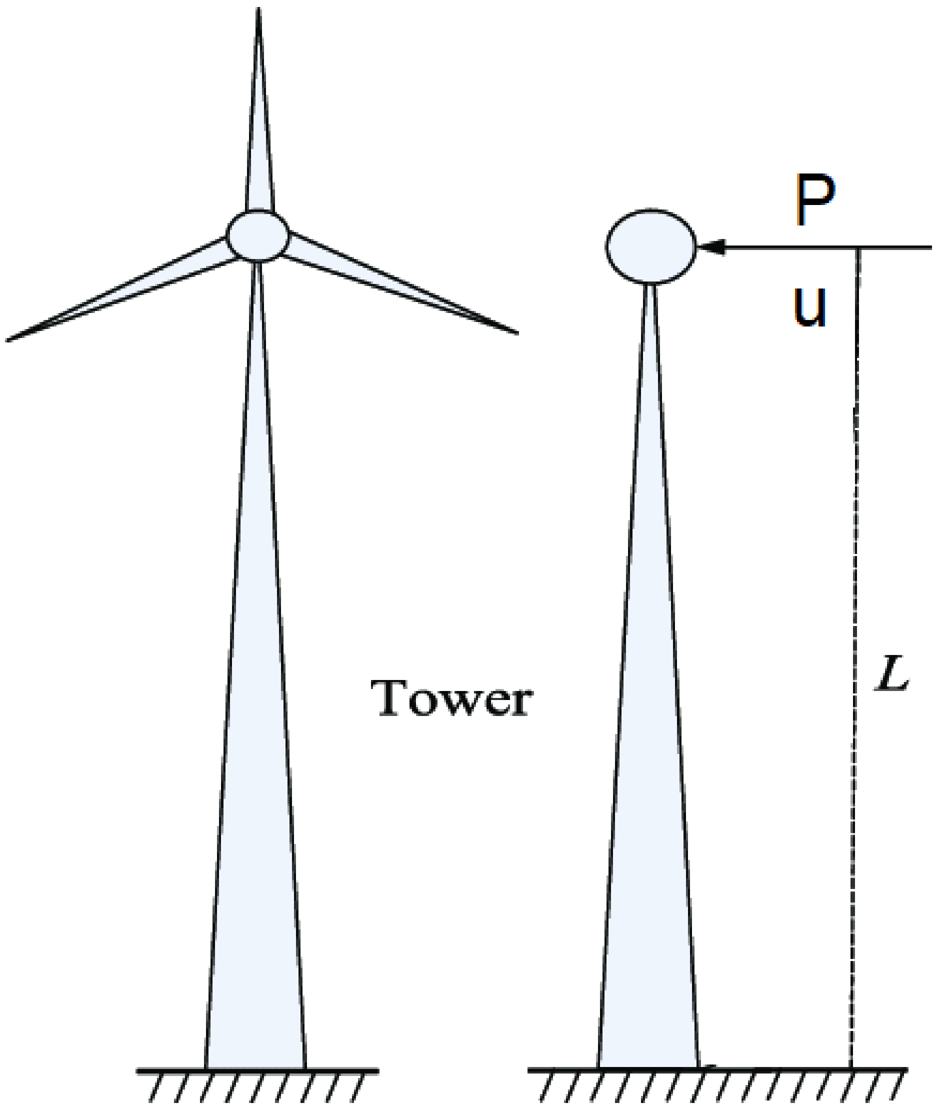
$$(k_0 + k_N)u = P$$

$$k_N = f(u)$$





$$k_N = f(u)$$



Direct Substitution

Let load P_A be applied to a softening spring.

Assume $k_N = 0$ for the first iteration.

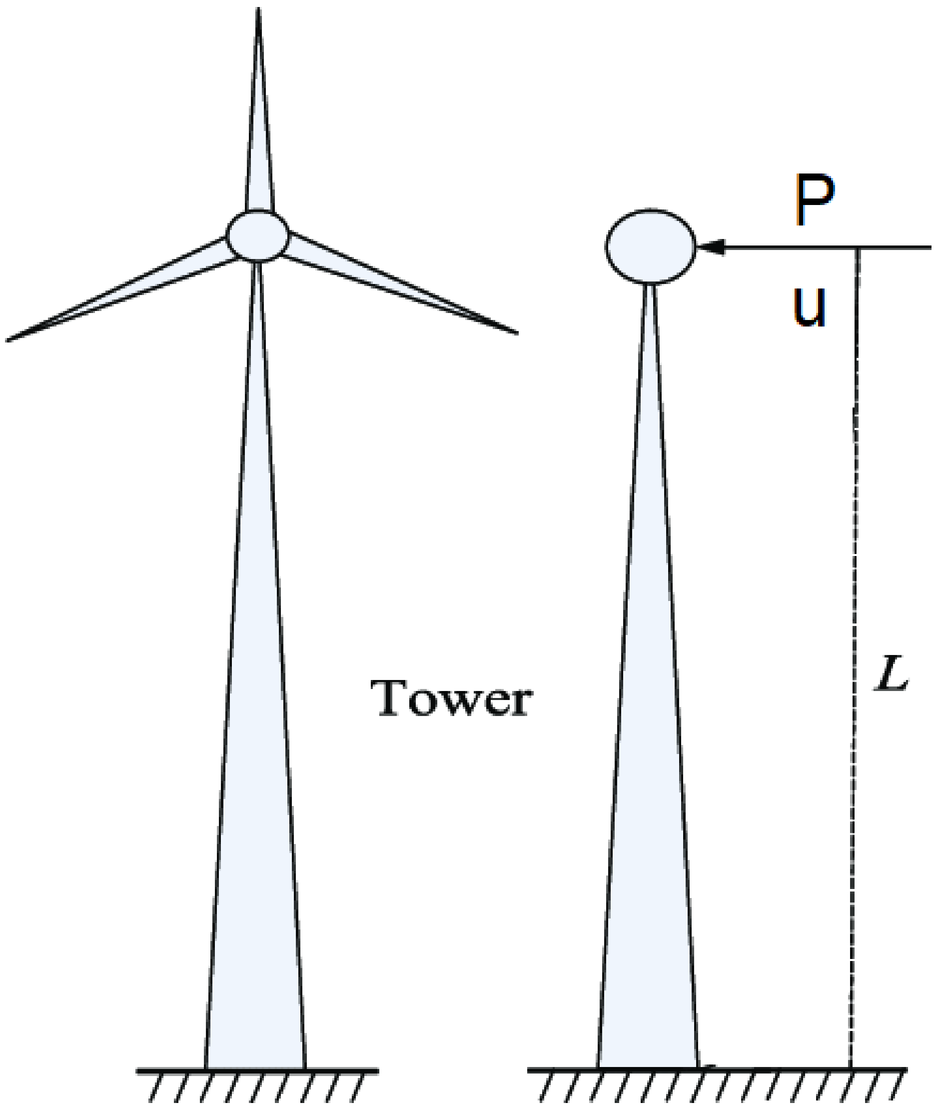
Compute first approximation to displacement:

$$u_1 = P_A/k_0$$

Use u_1 to compute new stiffness: $k = k_0 + f(u_1)$

Compute next approximation to displacement: $u_2 = P_A/k$

Generate sequence of approximations.



$$u_1 = k_0^{-1} P_A$$

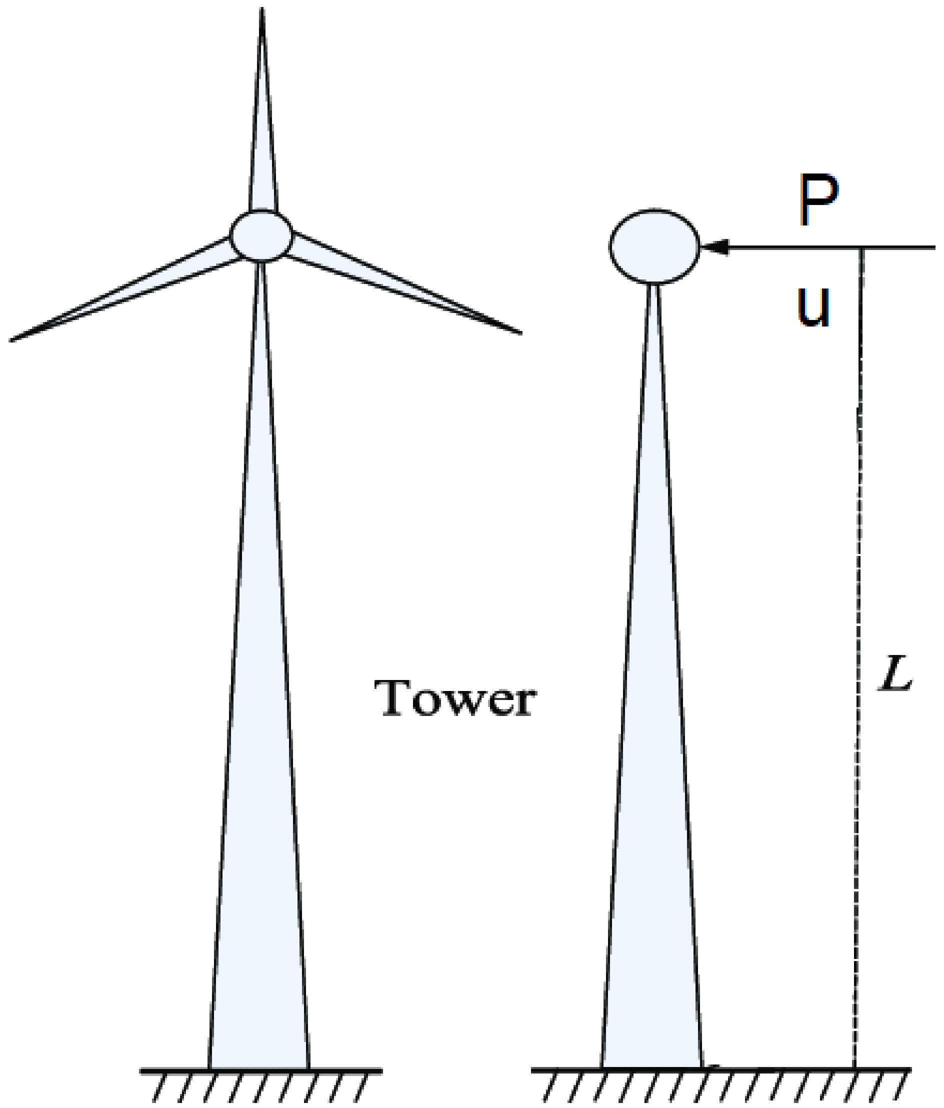
$$u_2 = \left(k_0 + k_{N1}\right)^{-1} P_A$$

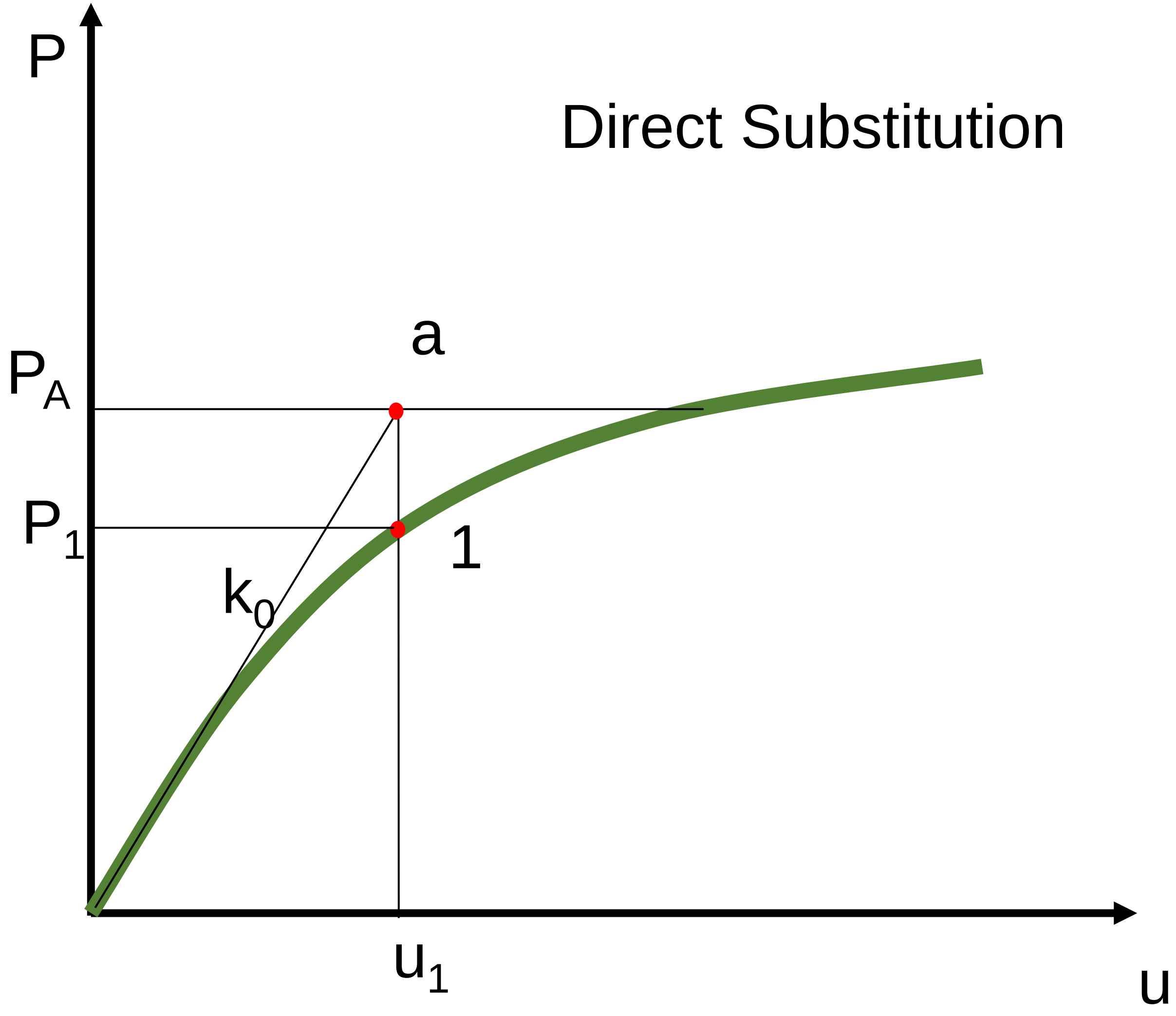
$$u_3 = \left(k_0 + k_{N2}\right)^{-1} P_A$$

$$u_4 = \left(k_0 + k_{N3}\right)^{-1} P_A$$

$$\vdots$$

$$u_{i+1} = \left(k_0 + k_{Ni}\right)^{-1} P_A$$





$$u_1 = k_0^{-1} P_A$$

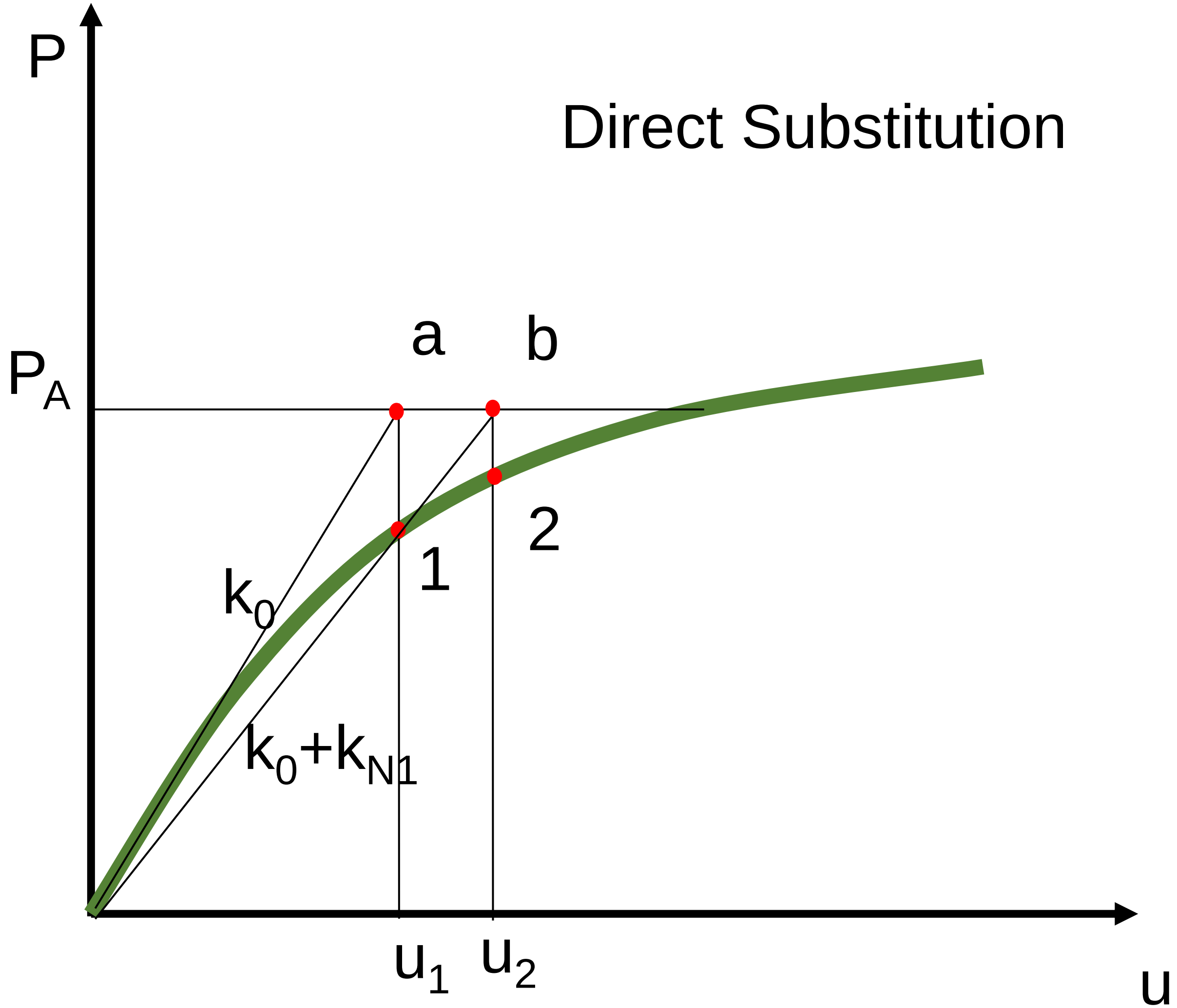
$$u_2 = \left(k_0 + k_{N1}\right)^{-1} P_A$$

$$u_3 = \left(k_0 + k_{N2}\right)^{-1} P_A$$

$$u_4 = \left(k_0 + k_{N3}\right)^{-1} P_A$$

$$\vdots$$

$$u_{i+1} = \left(k_0 + k_{Ni}\right)^{-1} P_A$$



$$u_1 = k_0^{-1} P_A$$

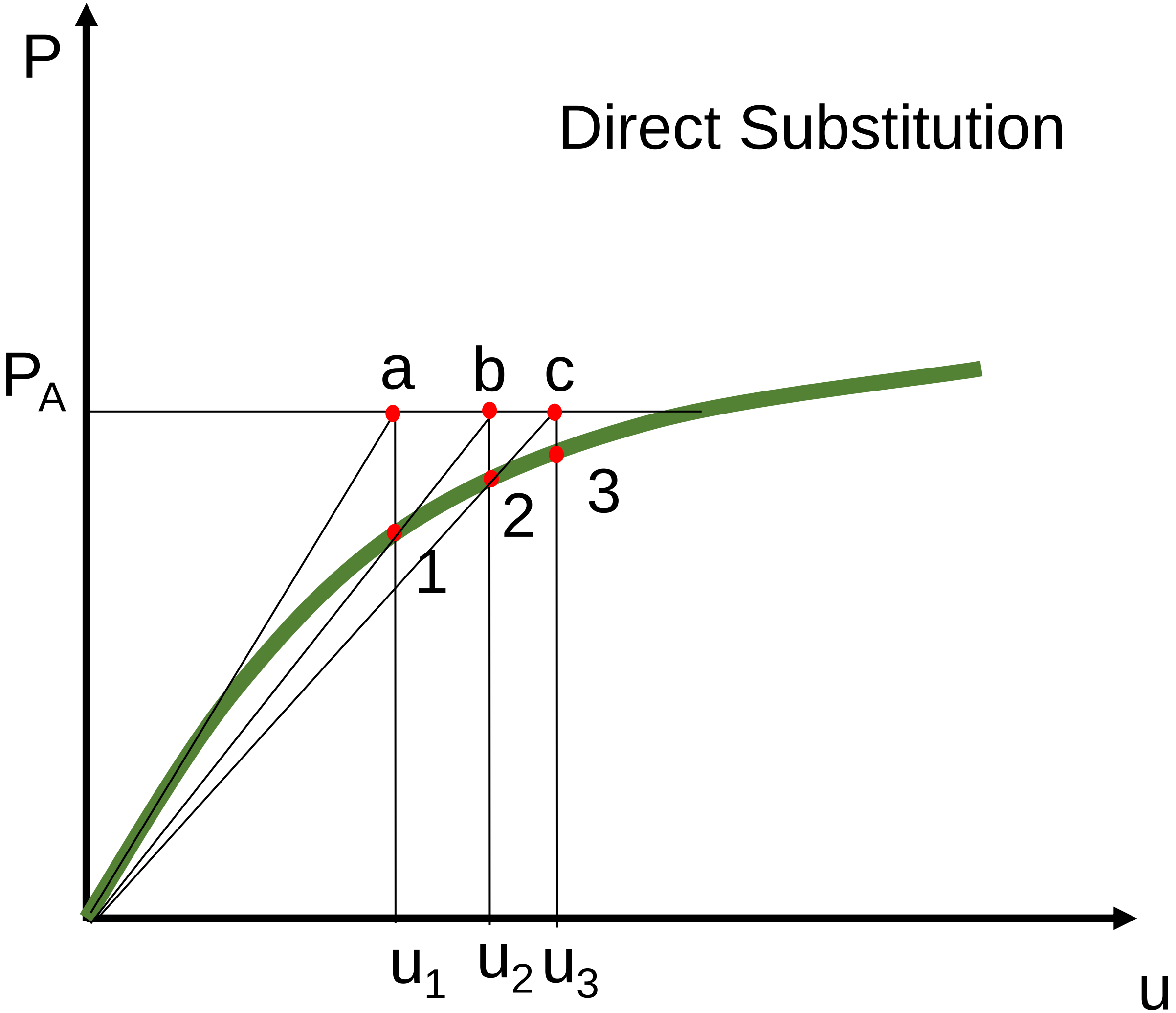
$$u_2 = \left(k_0 + k_{N1}\right)^{-1} P_A$$

$$u_3 = \left(k_0 + k_{N2}\right)^{-1} P_A$$

$$u_4 = \left(k_0 + k_{N3}\right)^{-1} P_A$$

$$\vdots$$

$$u_{i+1} = \left(k_0 + k_{Ni}\right)^{-1} P_A$$



$$u_1 = k_0^{-1} P_A$$

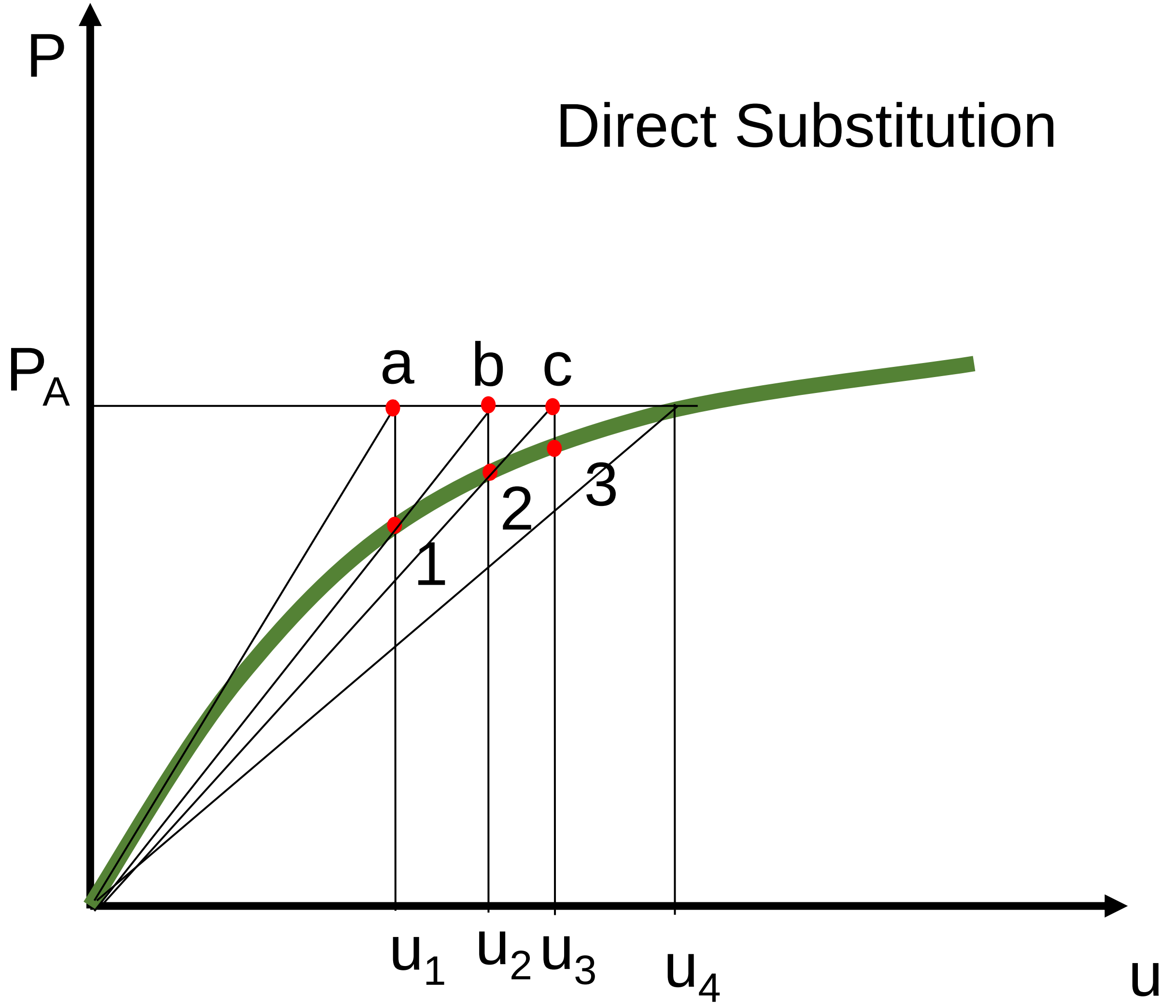
$$u_2 = \left(k_0 + k_{N1}\right)^{-1} P_A$$

$$u_3 = \left(k_0 + k_{N2}\right)^{-1} P_A$$

$$u_4 = \left(k_0 + k_{N3}\right)^{-1} P_A$$

$$\vdots$$

$$u_{i+1} = \left(k_0 + k_{Ni}\right)^{-1} P_A$$



$$u_1 = k_0^{-1} P_A$$

$$u_2 = \left(k_0 + k_{N1}\right)^{-1} P_A$$

$$u_3 = \left(k_0 + k_{N2}\right)^{-1} P_A$$

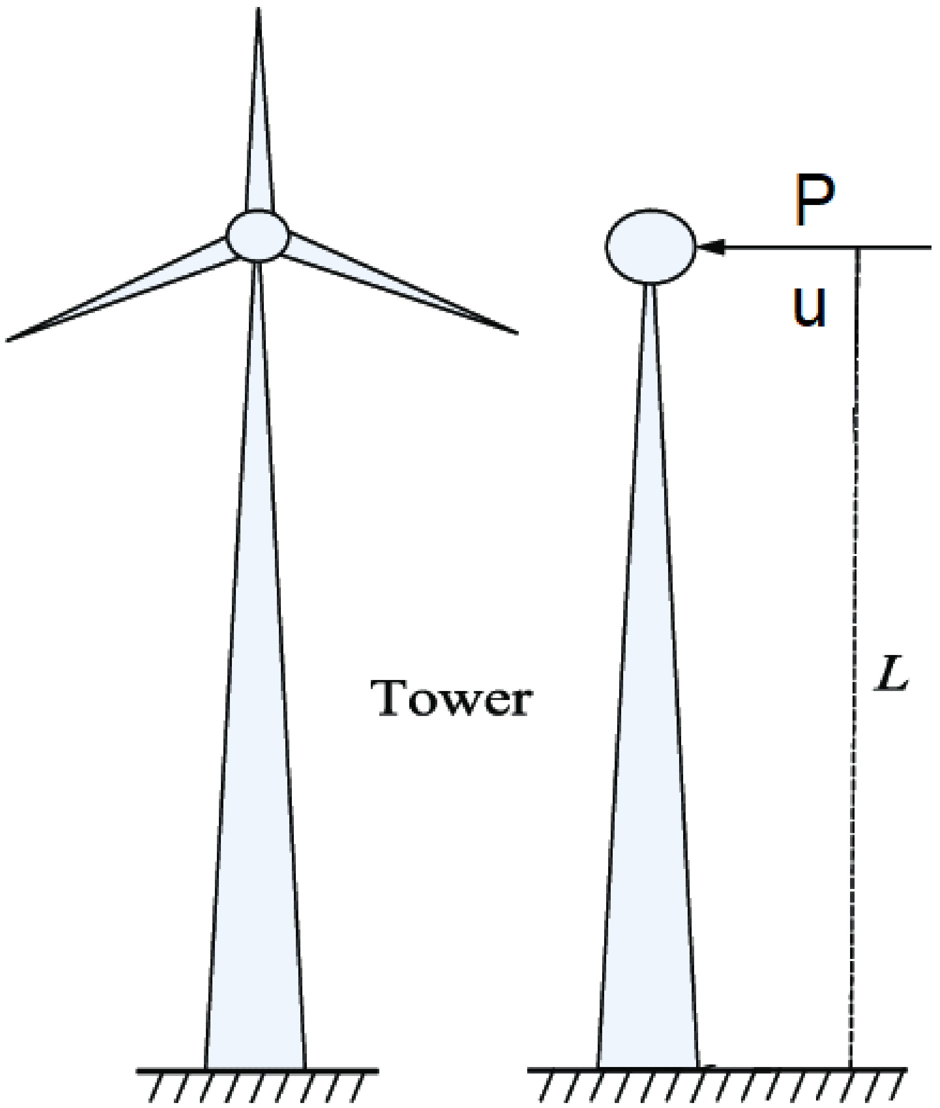
$$u_4 = \left(k_0 + k_{N3}\right)^{-1} P_A$$

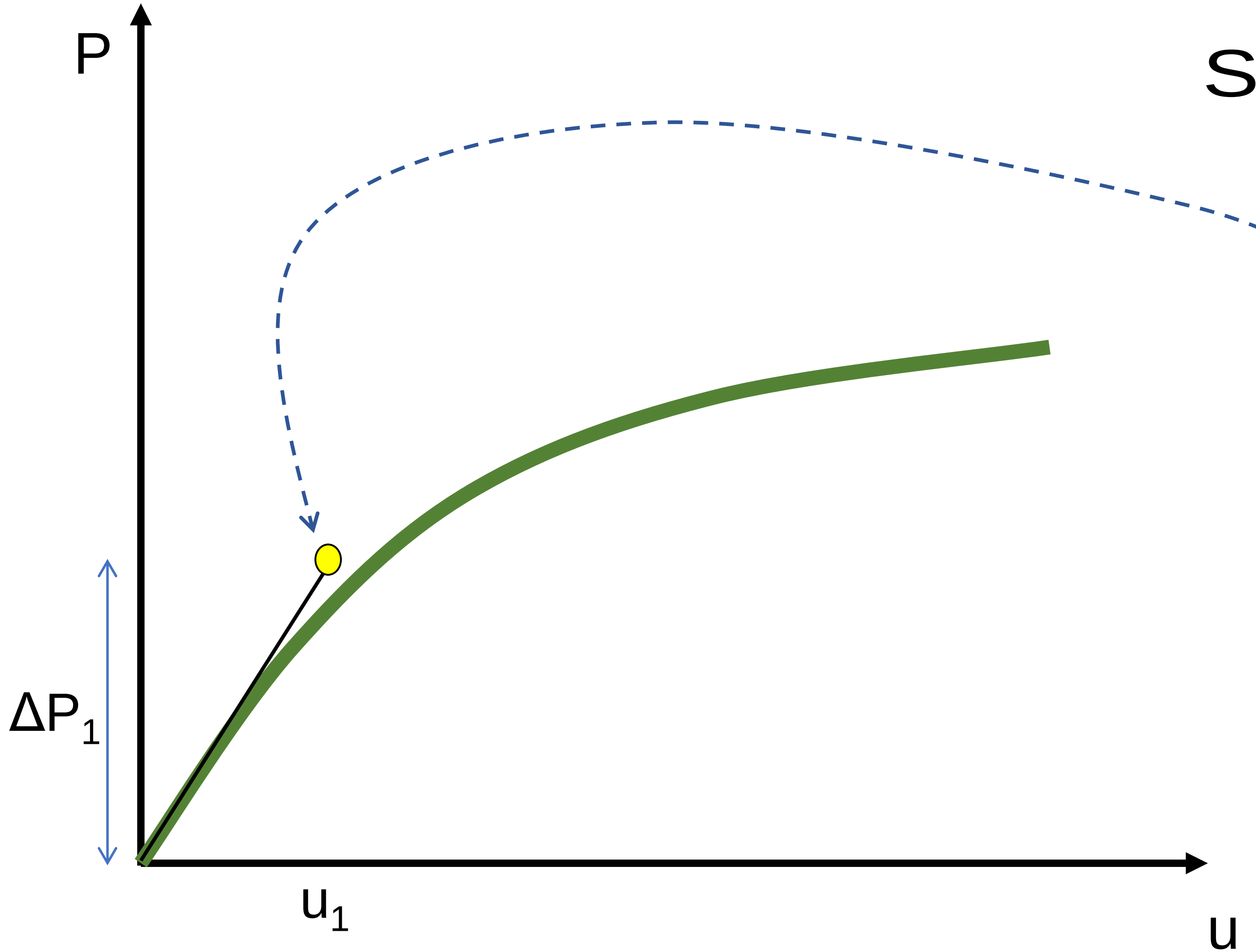
$$\vdots$$

$$u_{i+1} = \left(k_0 + k_{Ni}\right)^{-1} P_A$$

Incremental Approach

- Apply loads in a number of small increments.
- Iterate and Converge for each increment.
- Create entire load-displacement history.





Start at P = 0 and u = 0

Euler's Method

$$u_1 = 0 + (k_t)_0^{-1} \Delta P_1$$

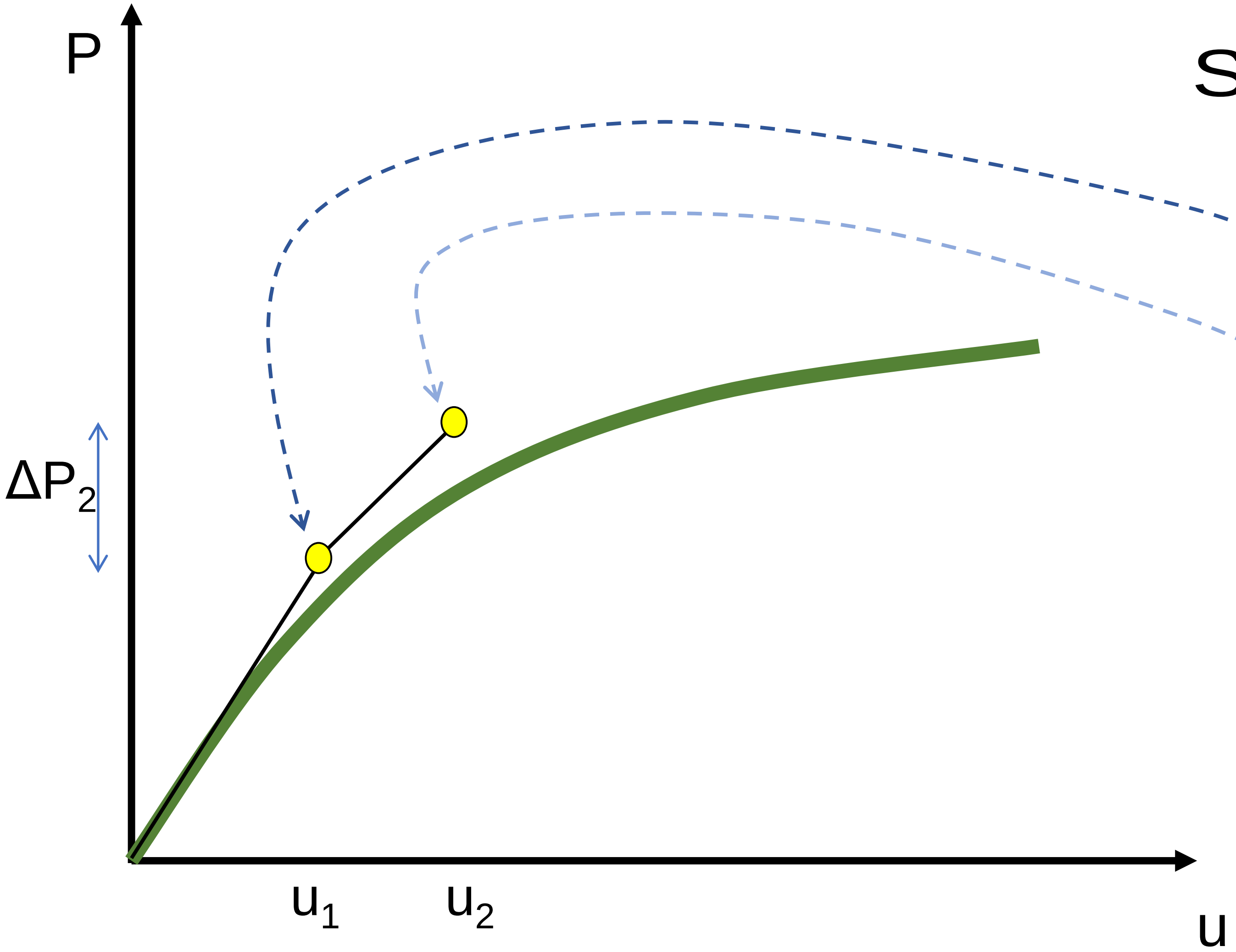
$$u_2 = u_1 + (k_t)_1^{-1} \Delta P_2$$

$$u_3 = u_2 + (k_t)_2^{-1} \Delta P_3$$

$$\vdots$$

$$u_{i+1} = u_i + (k_t)_i^{-1} \Delta P_{i+1}$$

Purely incremental approach with no corrections



Start at P = 0 and u = 0

Euler's Method

$$u_1 = 0 + (k_t)_0^{-1} \Delta P_1$$

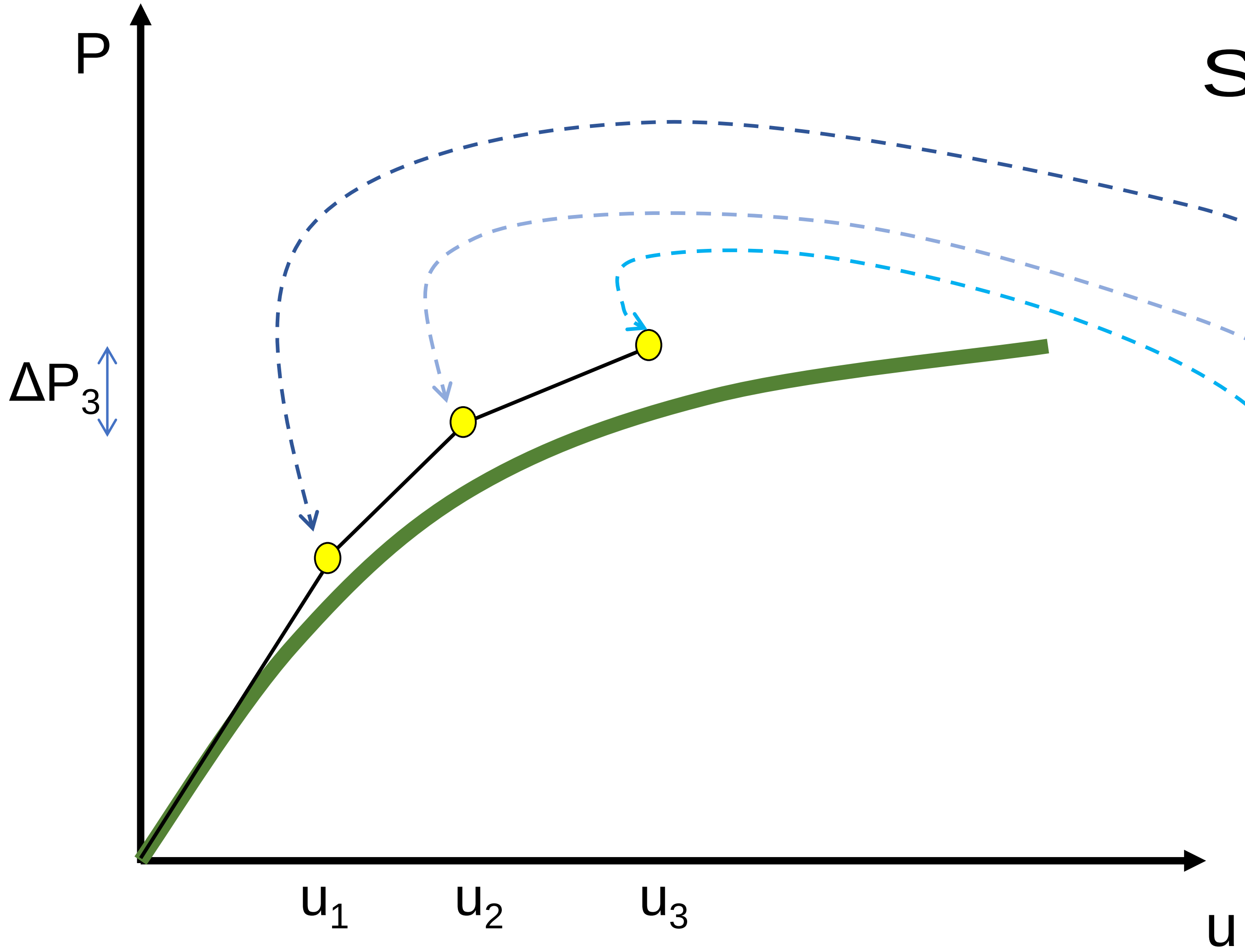
$$u_2 = u_1 + (k_t)_1^{-1} \Delta P_2$$

$$u_3 = u_2 + (k_t)_2^{-1} \Delta P_3$$

$$\vdots$$

$$u_{i+1} = u_i + (k_t)_i^{-1} \Delta P_{i+1}$$

Purely incremental approach with no corrections



Start at P = 0 and u = 0

Euler's Method

$$u_1 = 0 + (k_t)_0^{-1} \Delta P_1$$

$$u_2 = u_1 + (k_t)_1^{-1} \Delta P_2$$

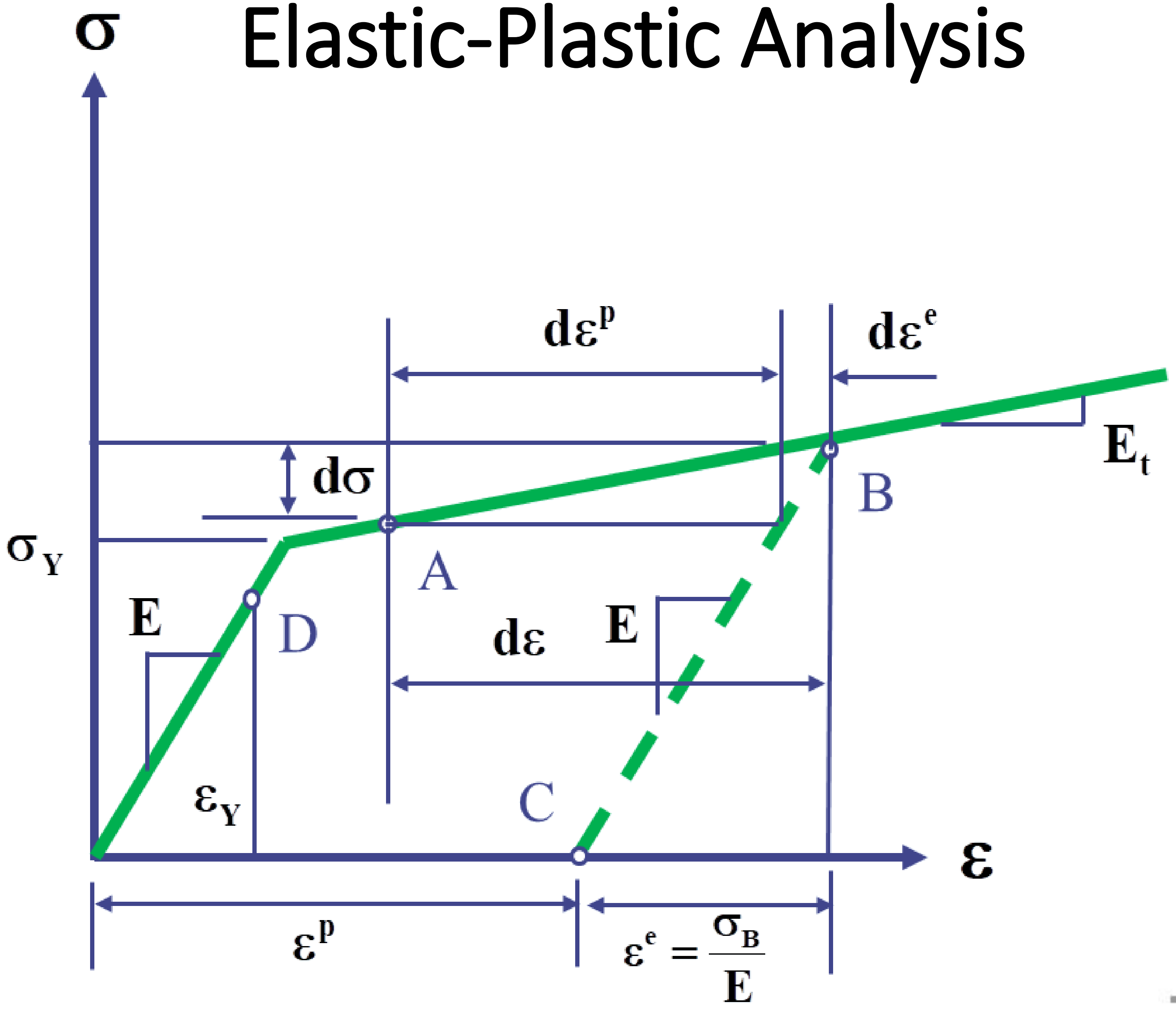
$$u_3 = u_2 + (k_t)_2^{-1} \Delta P_3$$

$$\vdots$$

$$u_{i+1} = u_i + (k_t)_i^{-1} \Delta P_{i+1}$$

Purely incremental approach with no corrections

Elastic-Plastic Analysis



- Yielding has occurred.
- Strain increment dε takes place.
- $d\varepsilon = d\varepsilon^e + d\varepsilon^p$
- Write stress increment in various ways:

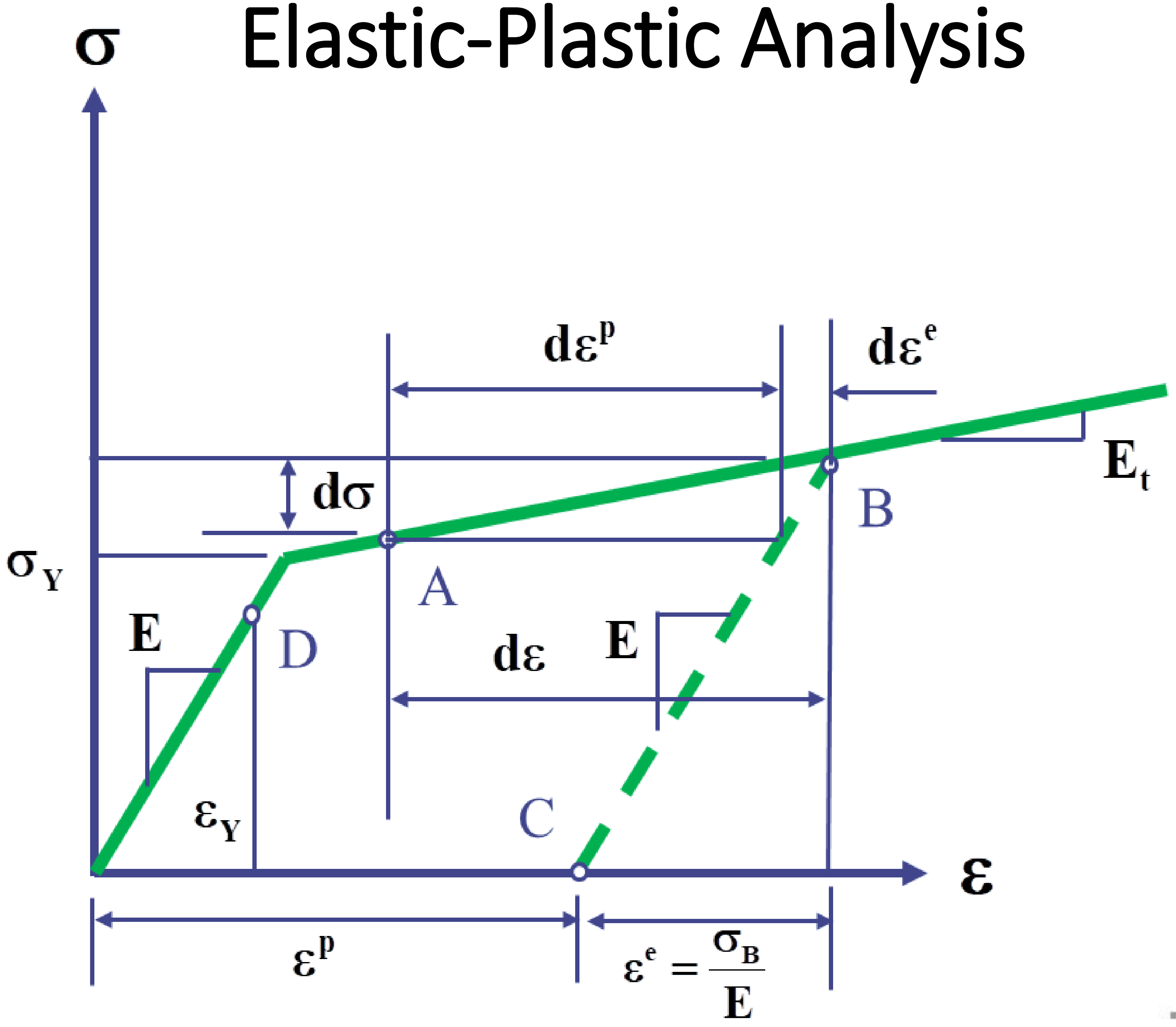
Plastic Flow

$$d\sigma = E\left(d\varepsilon - d\varepsilon^p\right)$$

$$d\sigma = E_t d\varepsilon$$

$$d\sigma = H d\varepsilon^p$$

Elastic-Plastic Analysis



$$d\sigma = E\left(d\varepsilon - d\varepsilon^{p}\right)$$

$$d\sigma = E_t d\varepsilon$$

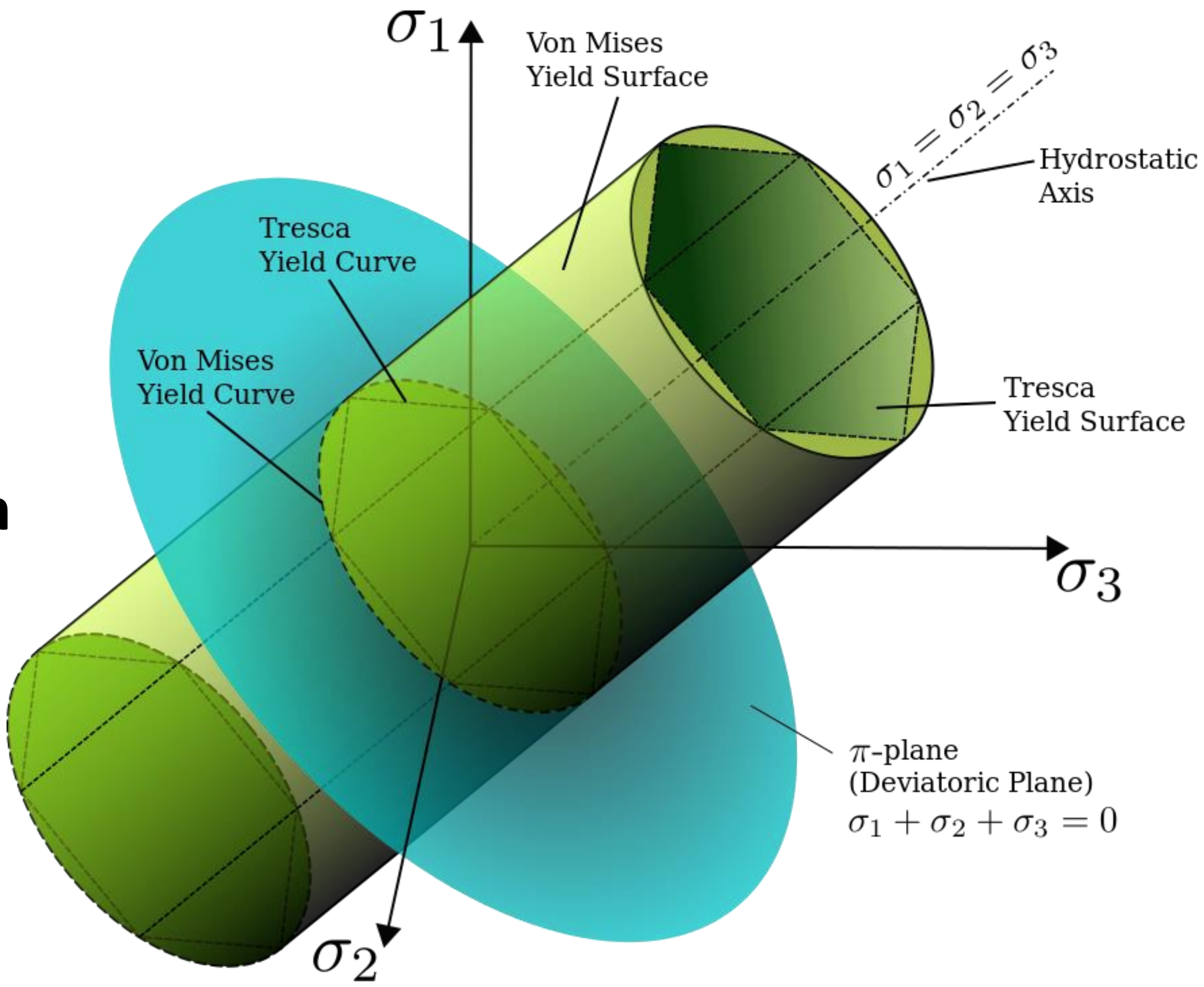
$$d\sigma = H d\varepsilon^{p}$$

$$H = \left(\frac{E_t}{1-\left(E_t / E\right)}\right)$$

$$E_t = E\left(1-\frac{E}{E+H}\right)$$

Yield Criterion

- **Defines the onset of yielding**
- **$|\sigma| = \sigma_y$**
- **σ_y - yield stress in uniaxial tension**
- **Tresca**
- **von Mises**

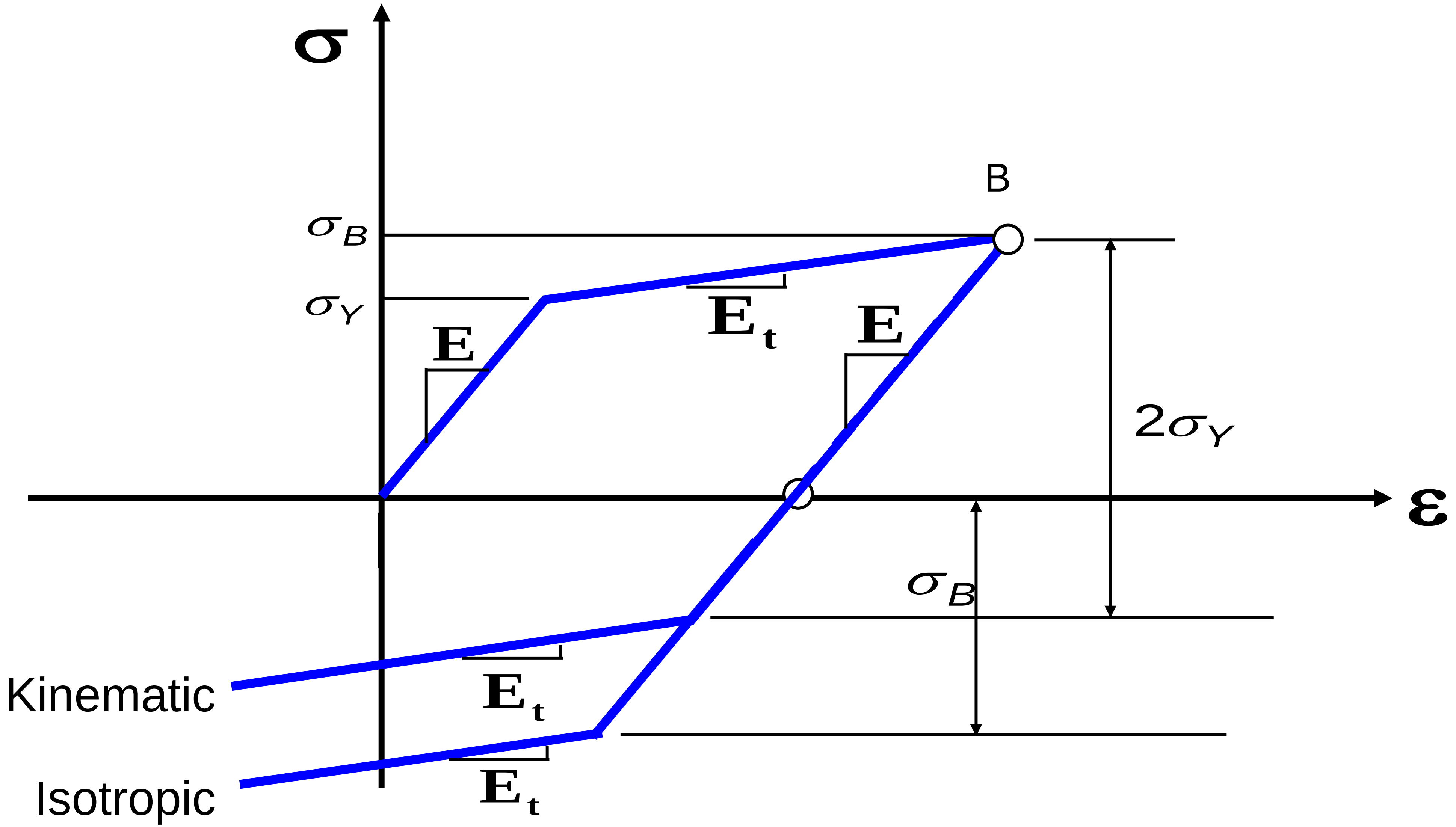


Flow Rule

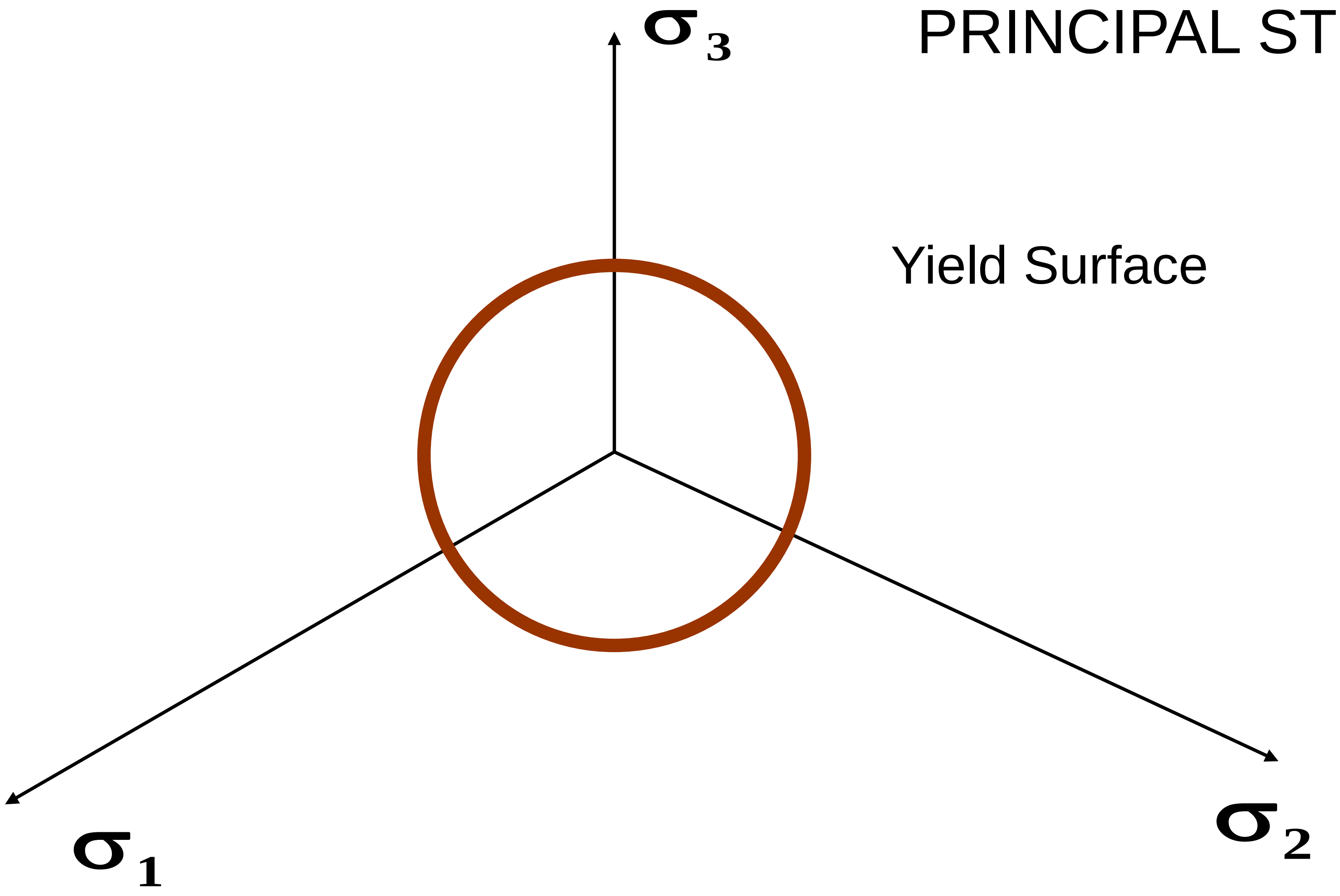
- Relates stress increment {$d\sigma$} to strain increment {$d\varepsilon$} after yielding.
- Uniaxial case: $d\sigma = E_t \, d\varepsilon$
- Prandtl-Reuss often used.
- Associated - ductile materials.
- Nonassociated - soil or granular materials.

Hardening Rules

- Kinematic
  - Yield surface retains size and shape and translates in stress space.
- Isotropic
  - Yield surface retains shape but increases in size.

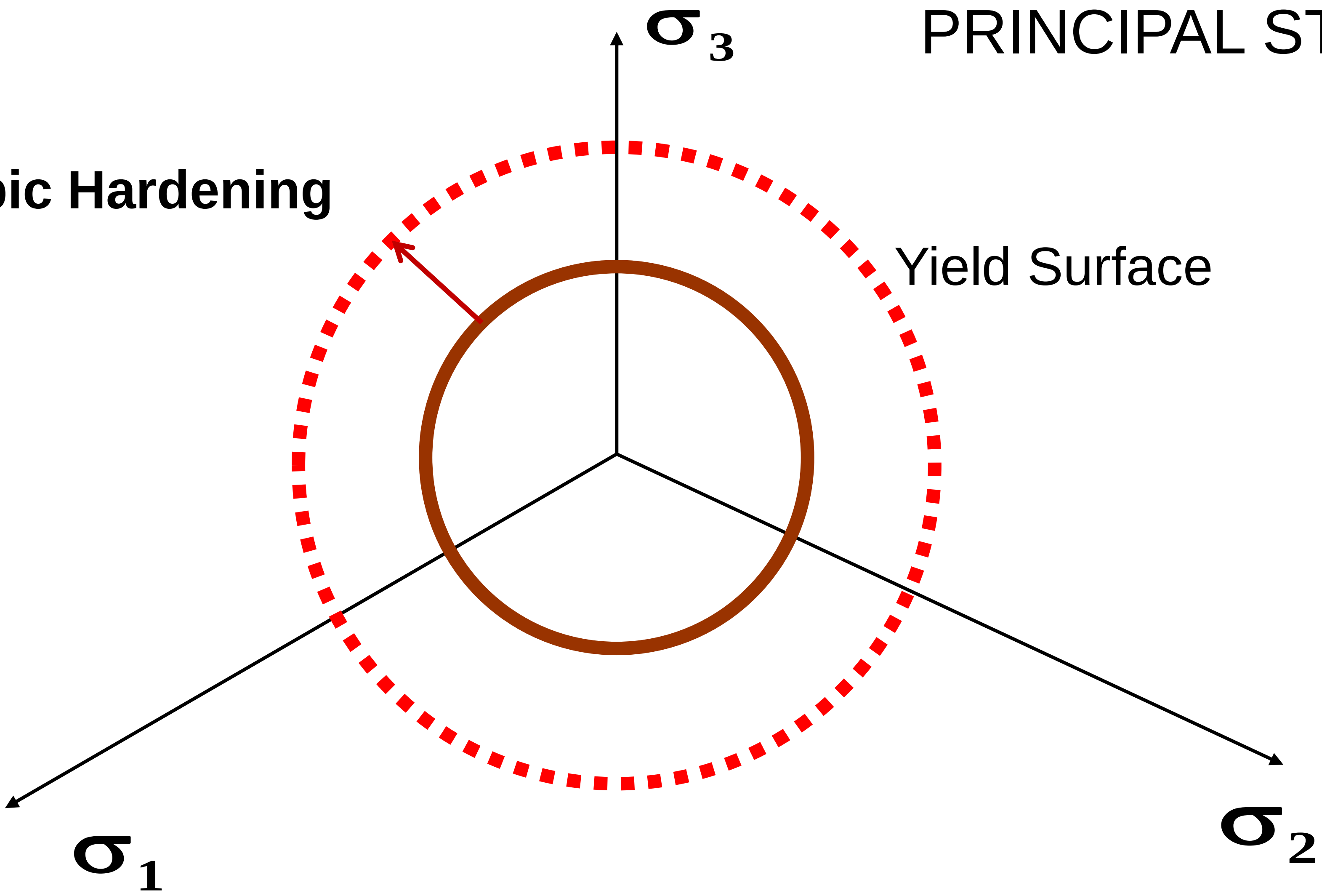

σ
B
σ_B
σ_Y
E
E_t
E
$2\sigma_Y$
ε
σ_B
Kinematic
E_t
Isotropic
E_t

PRINCIPAL STRESS SPACE



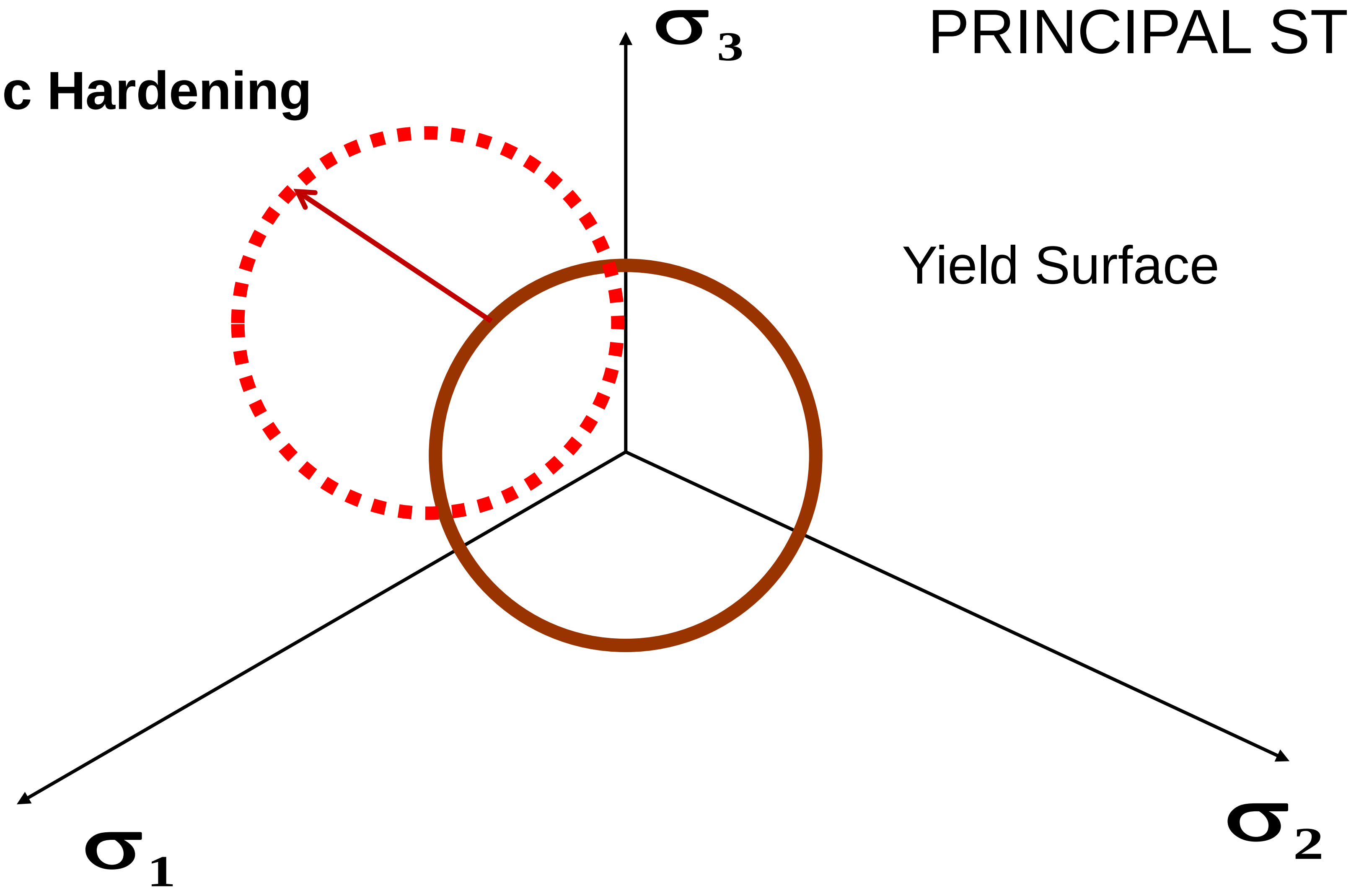
PRINCIPAL STRESS SPACE

Isotropic Hardening



PRINCIPAL STRESS SPACE

Kinematic Hardening



Multi-Dimensional Plasticity

- Simplest model: total strain is sum of elastic and plastic parts: $\varepsilon=\varepsilon_e+\varepsilon_p$
- Stress only depends on elastic part (so rest state includes plastic strain): $\sigma=\sigma(\varepsilon_e)$
- If σ is too big, we yield, and transfer some of ε_e into ε_p so that σ is acceptably small

Multi-Dimensional Yield criteria

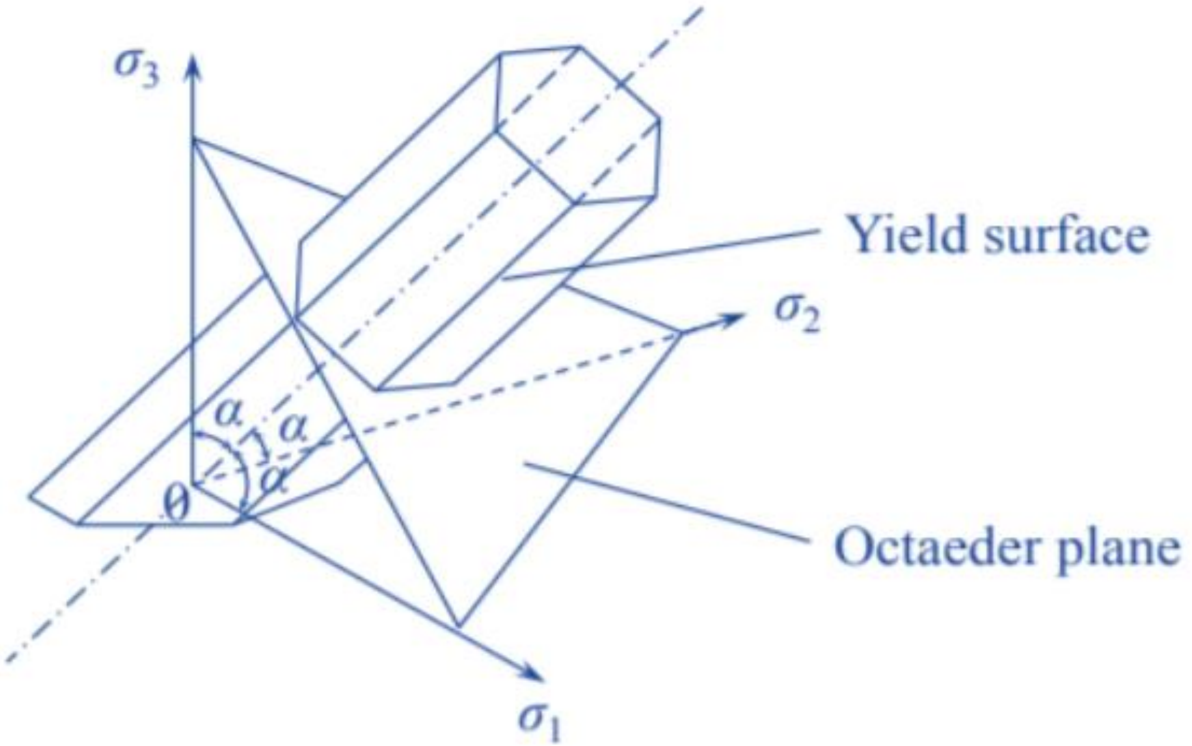
- Lots of complicated stuff happens when materials yield
  - Metals (e.g. tower): dislocations moving around
  - Polymers (e.g. blades): molecules sliding against each other
  - Granular materials (e.g. soil): plasticity and consolidation
- Difficult to characterize exactly when plasticity (yielding) starts
  - Work hardening etc. mean it changes all the time too
- Approximations needed
  - Tresca and Von Mises

Yielding

- First note that shear stress is the important quantity
- Materials (almost) never can permanently change their volume
- Plasticity should ignore volume-changing stress

Tresca yield criterion

The Tresca criterion is equivalent to saying that yielding will occur at a critical value of the maximum shear stress, consistent with micromechanical behavior of crystals, involving slip and dislocation motion.



$$\tau_o = \max\left\{\frac{|\sigma_1-\sigma_2|}{2}, \frac{|\sigma_2-\sigma_3|}{2}, \frac{|\sigma_3-\sigma_1|}{2}\right\}$$

$$\max\{|\sigma_1-\sigma_2|, |\sigma_2-\sigma_3|, |\sigma_3-\sigma_1|\} = \sigma_y$$

Tresca yield criterion

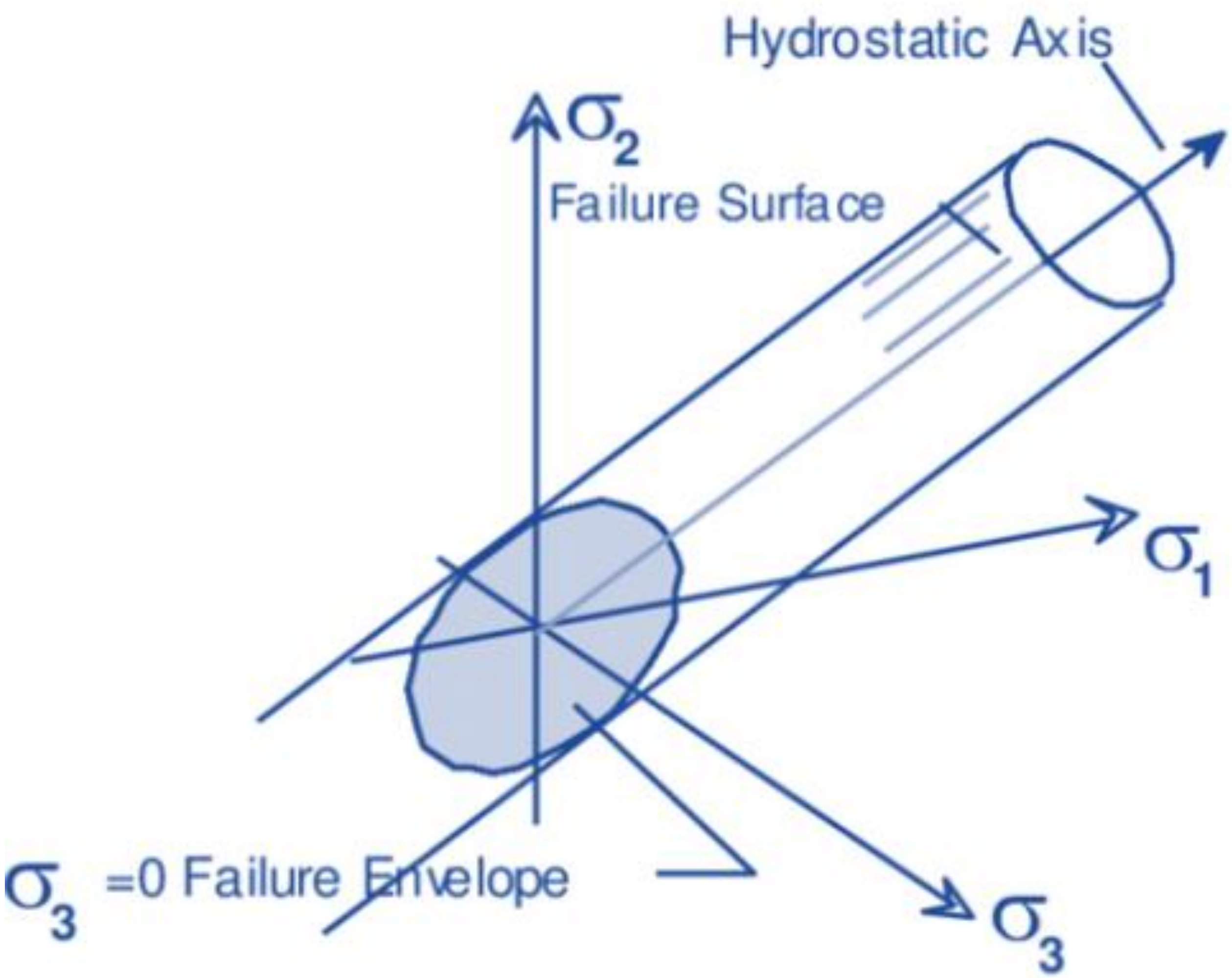
- This is the simplest description:
  - Change basis to diagonalize σ
  - Look at normal stresses (i.e. the eigenvalues of σ)
  - **No yield if $\sigma_{max}-\sigma_{min} \leq \sigma_Y$**
- Tends to be conservative (rarely predicts yielding when it shouldn’t happen)
- But, not so accurate for some stress states
  - Doesn’t depend on middle normal stress at all
- Big problem (mathematically): not smooth

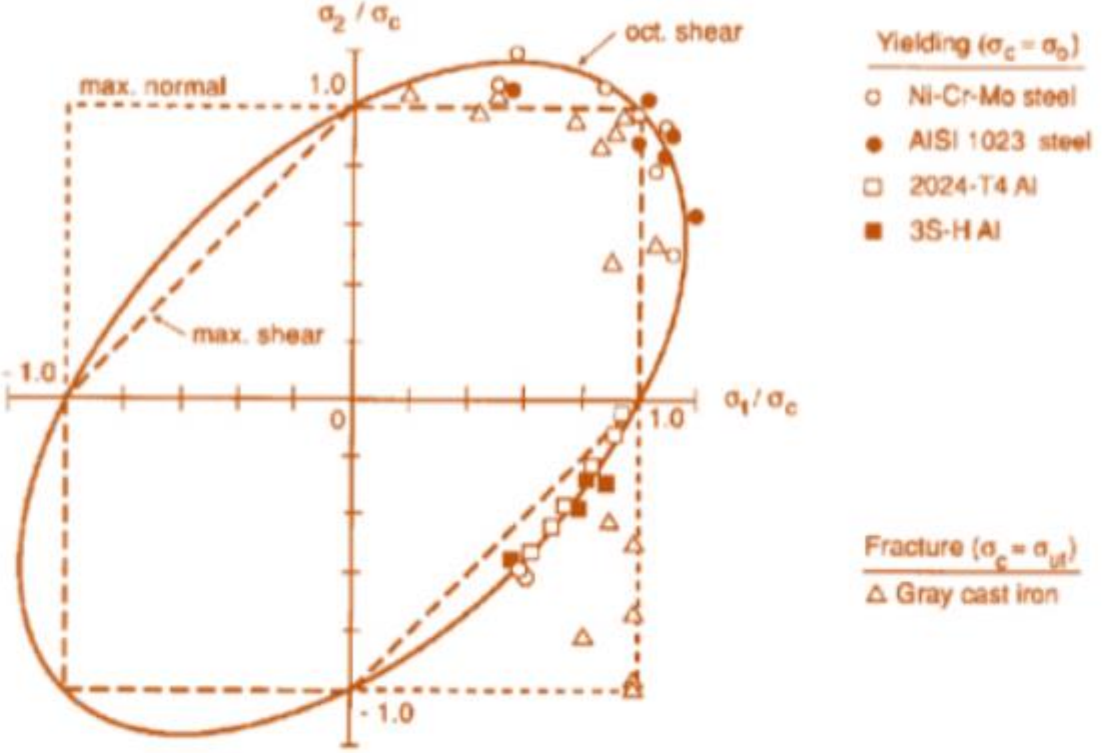
Von Mises yield criterion

The von Mises criterion states that failure occurs when the energy of distortion reaches the same energy for yield/failure in uniaxial tension.

State of stress	Boundary conditions	von Mises equations
General	No restrictions	$\sigma_v = \sqrt{\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2\right]+3\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right)}$
Principal stresses	$\sigma_{12}=\sigma_{31}=\sigma_{23}=0$	$\sigma_v = \sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]}$
General plane stress	$\sigma_3=0$ $\sigma_{31}=\sigma_{23}=0$	$\sigma_v = \sqrt{\sigma_{11}^2-\sigma_{11}\sigma_{22}+\sigma_{22}^2+3\sigma_{12}^2}$
Principal plane stress	$\sigma_3=0$ $\sigma_{12}=\sigma_{31}=\sigma_{23}=0$	$\sigma_v = \sqrt{\sigma_1^2-\sigma_1\sigma_2+\sigma_2^2}$
Pure shear	$\sigma_1=\sigma_2=\sigma_3=0$ $\sigma_{31}=\sigma_{23}=0$	$\sigma_v = \sqrt{3}\lvert\sigma_{12}\rvert$
Uniaxial	$\sigma_2=\sigma_3=0$ $\sigma_{12}=\sigma_{31}=\sigma_{23}=0$	$\sigma_v = \sigma_1$

Von Mises yield criterion





Perfect plastic flow

- Once yield condition says so, need to start changing plastic strain
- The magnitude of the change of plastic strain should be such that we stay on the yield surface
  - I.e. maintain f(σ)=0
    (where f(σ)≤0 is, say, the von Mises condition)
- The direction that plastic strain changes isn't as straightforward
- "Associative" plasticity: $$\dot{\varepsilon}_p = \gamma \frac{\partial f}{\partial \sigma}$$

Algorithm

After a time step, check von Mises criterion:

is $f(\sigma) = \sqrt{\frac{3}{2}}\|dev(\sigma)\|_F - \sigma_Y > 0$

If so, need to update plastic strain:

$$\varepsilon_p^{new} = \varepsilon_p + \gamma \frac{\partial f}{\partial \sigma} = \varepsilon_p + \gamma \sqrt{\frac{3}{2}} \frac{dev(\sigma)}{\|dev(\sigma)\|_F}$$

with γ chosen so that $f(\sigma^{new})=0$ (easy for linear elasticity)

Introduction to Nonlinear Boundary Elements and Finite Elements

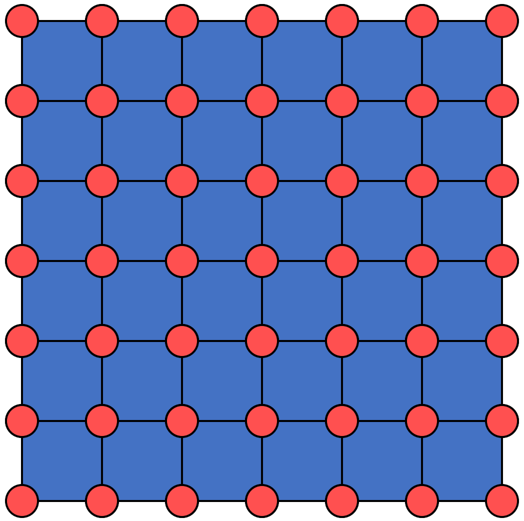
Boundary Element Method – BEM

- The boundary element method (BEM) also known as the boundary integral equation method (BIEM) is now firmly established in many engineering disciplines.
- The attraction of the method can be largely attributed to the reduction in the dimensionality of the problem; for two-dimensional (2D) plate and shell analyses only the line boundary of the domain needs to be discretized into elements and, for 3D problems, only the surface of the domain needs to be discretized.
- This means that, compared to domain type analysis, a boundary analysis results in a substantial reduction in data preparation and a much smaller system of equations to be solved. Furthermore, this simpler representation of the body means that regions of high stress concentration can be modeled more efficiently as the necessary high concentration of grid points is confined to one less dimension.

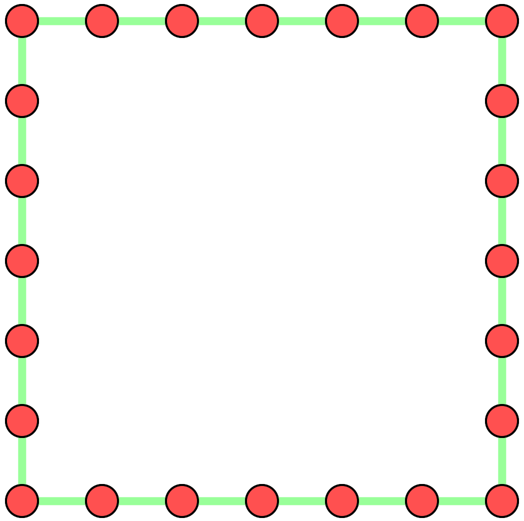
Finite Element Method – FEM

The finite element method (FEM) is a widely used method for numerically solving differential equations arising in engineering and mathematical modeling. Typical problem areas of interest include the traditional fields of structural analysis, heat transfer, fluid flow, mass transport, and electromagnetic potential. The FEM is a general numerical method for solving partial differential equations in two or three space variables (i.e., some boundary value problems). To solve a problem, the FEM subdivides a large system into smaller, simpler parts that are called finite elements. This is achieved by a particular space discretization in the space dimensions, which is implemented by the construction of a mesh of the object: the numerical domain for the solution, which has a finite number of points.

The basic difference FEM vs. BEM

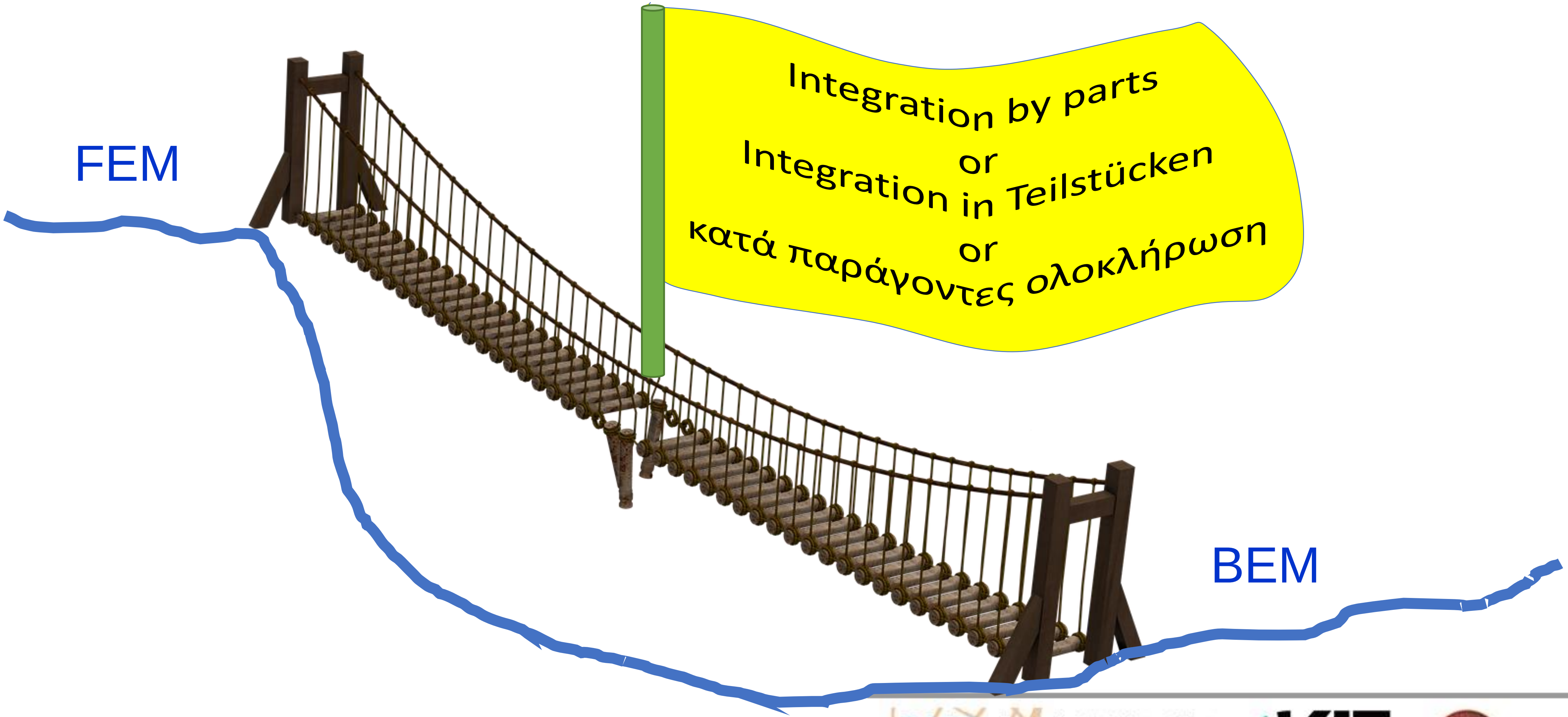


domain discretization
(finite elements)



boundary discretization
(boundary elements)

The bridge between FEM and BEM



References

- Bathe, K. J. (2006). Finite element procedures. Klaus-Jurgen Bathe.
- Hughes, T. J. (2012). The finite element method: linear static and dynamic finite element analysis. Courier Corporation.
- Gaul, L., Kögl, M., & Wagner, M. (2013). Boundary element methods for engineers and scientists: an introductory course with advanced topics. Springer Science & Business Media.
- Beer, G., Smith, I., & Duenser, C. (2008). The boundary element method with programming: for engineers and scientists. Springer Science & Business Media.

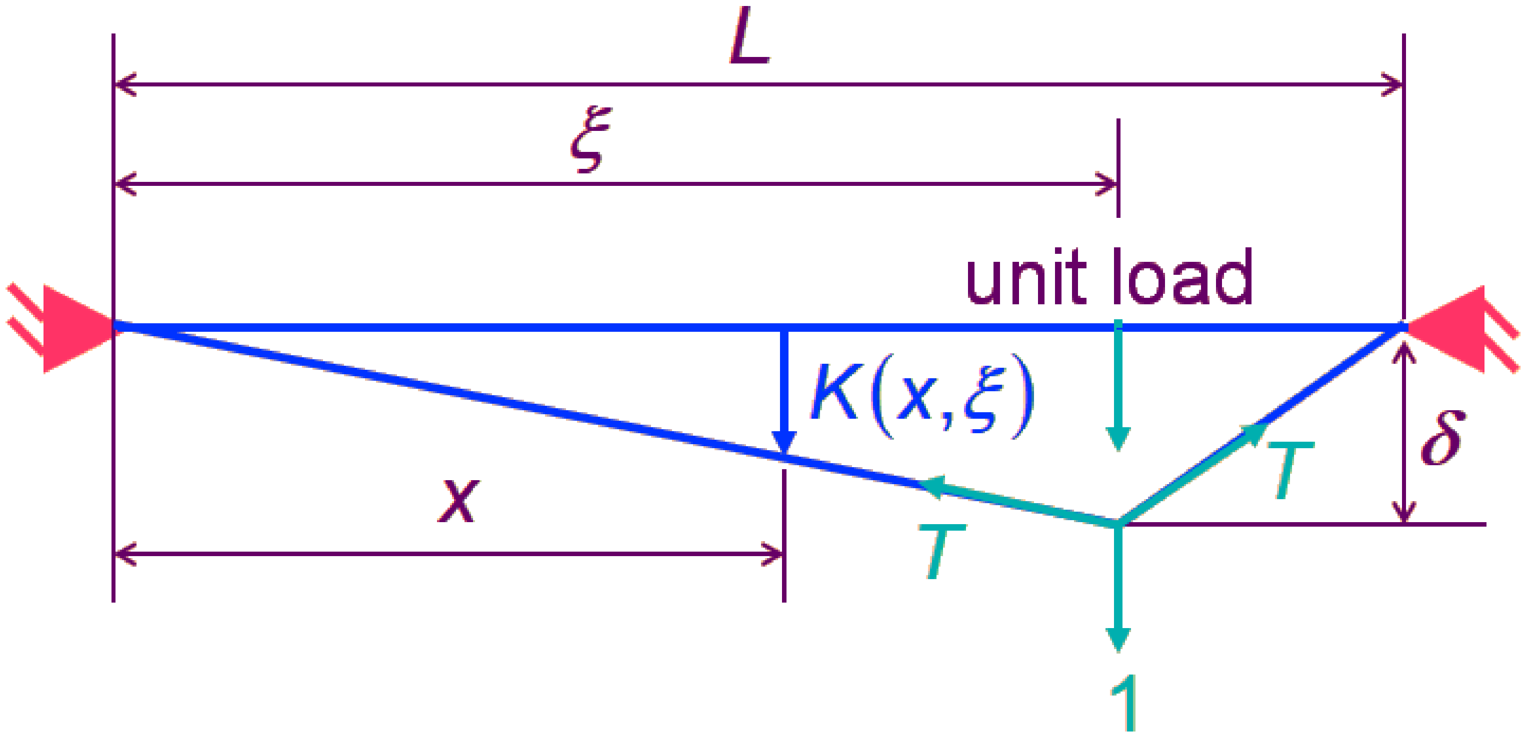
What is an Integral Equation?

- They are equations which contain the unknown function under the integral sign.
- For example,

$$f(x) = \int_a^b K(x,\xi)\phi(\xi)d\xi$$

known (function) — known (kernel) — unknown (function)

Application: String with Point Load using BEM

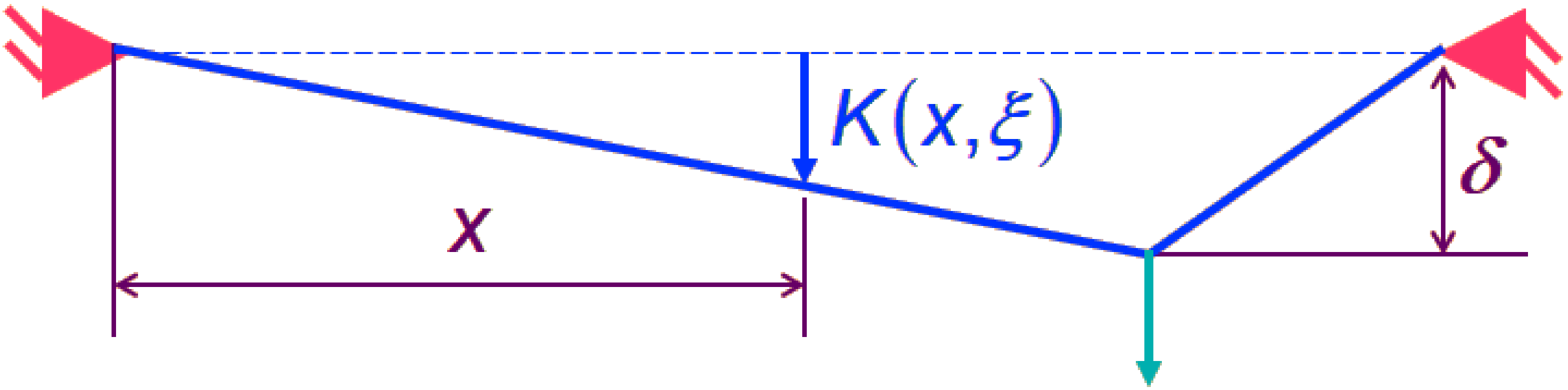


For small deflection, force equilibrium gives

$$T\frac{\delta}{\xi}+T\frac{\delta}{L-\xi}=1$$

Application: String with Point Load using BEM

From $T\dfrac{\delta}{\xi} + T\dfrac{\delta}{L-\xi} = 1$ we have $\delta = \dfrac{\xi(L-\xi)}{TL}$



For $0 \le x \le \xi$, $K(x,\xi) = \dfrac{x}{\xi}\dfrac{\xi(L-\xi)}{TL} = \dfrac{x(L-\xi)}{TL}$

For $0 \le x \le \xi$, $K(x,\xi) = \dfrac{(L-x)}{(L-\xi)}\dfrac{\xi(L-\xi)}{TL} = \dfrac{\xi(L-x)}{TL}$

Application: String with Point Load using BEM

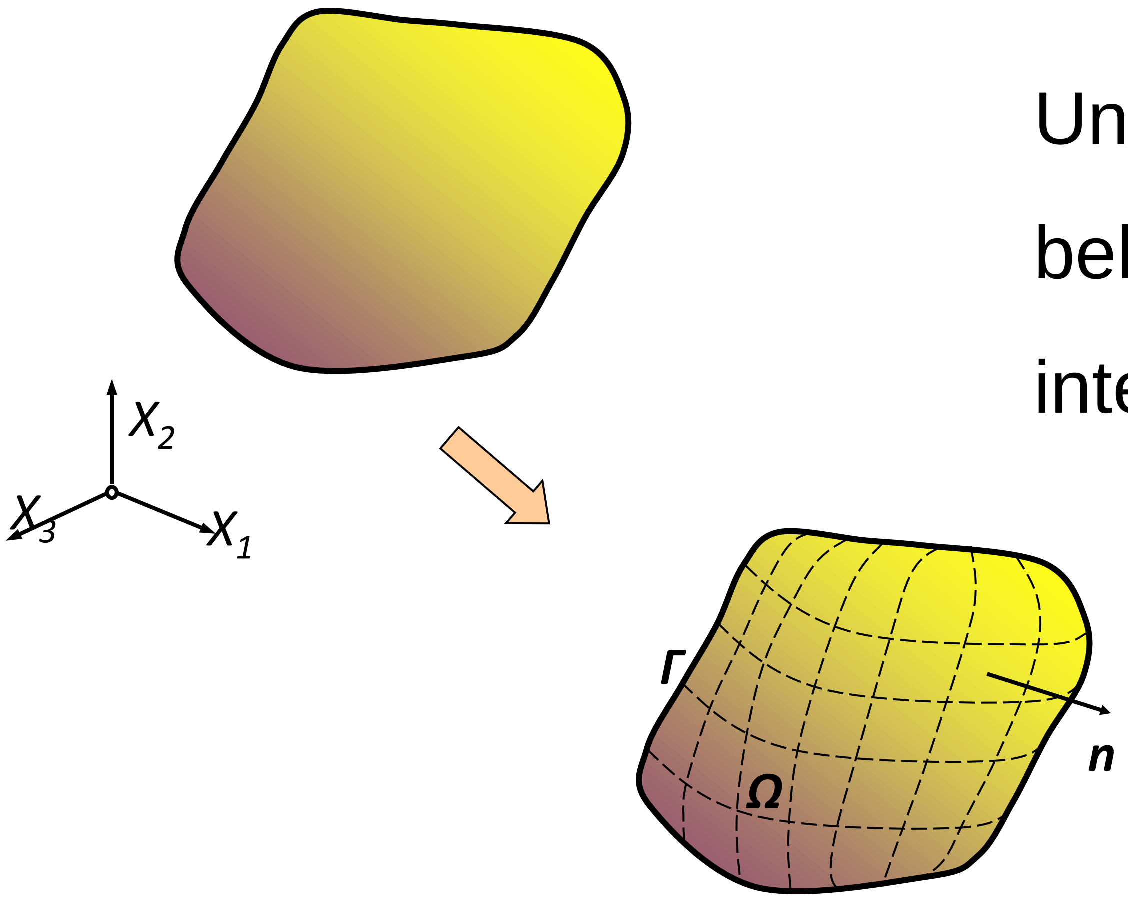
For small deflection, the Principle of Superposition holds, and for an arbitrary load density $\sigma(x)$,

the deflection $y(x)$ at any point is given by

$$y(x) = \int_0^L K(x,\xi)\sigma(\xi)d\xi$$

If we specify the deflection function $y(x)$, then this is an Integral Equation for the unknown load density function $\sigma(x)$.

Extension: 3-D dynamic inelastic problems - BEM



Unfortunately, due to inelastic behavior and inertia terms, an internal discretization is required.

Extension: 3-D dynamic inelastic problems - BEM

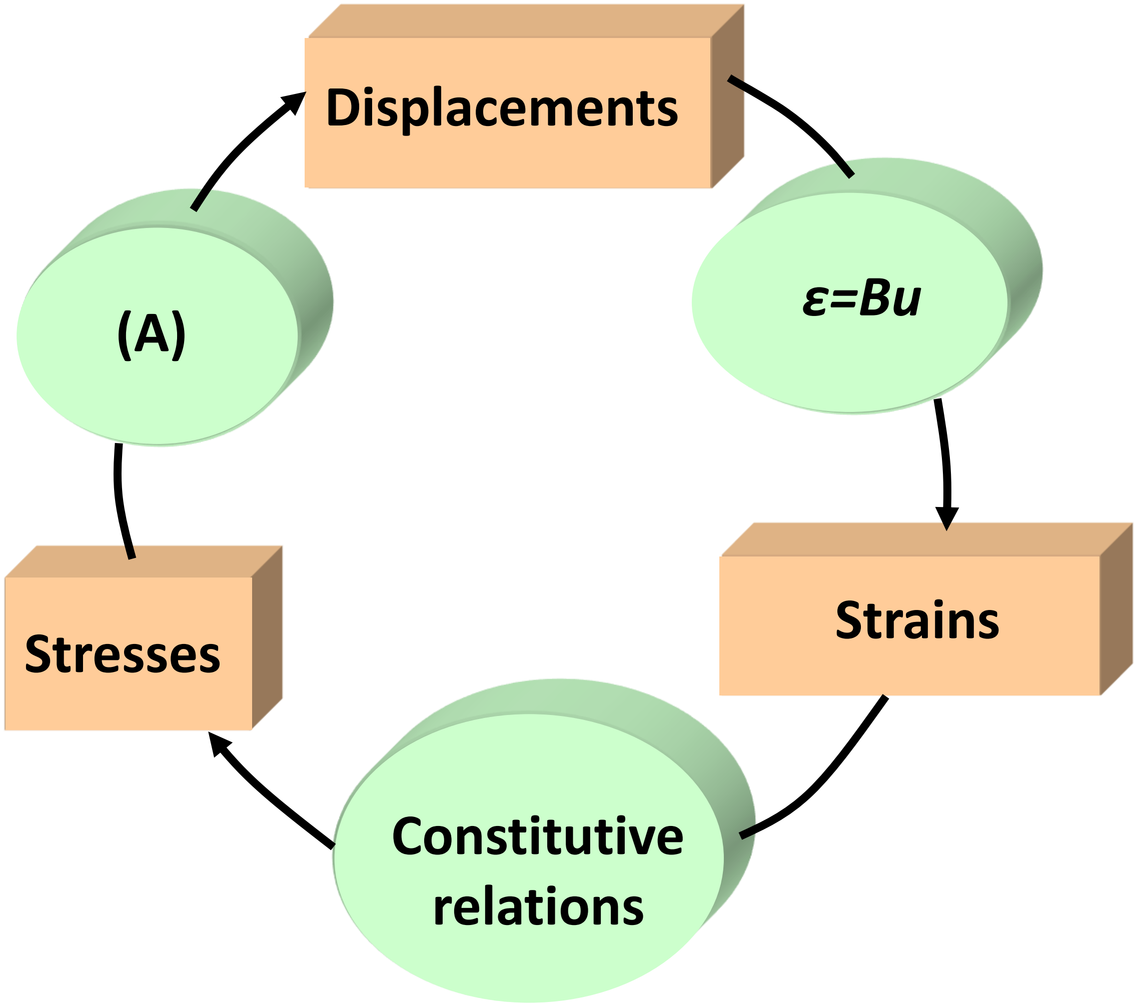
$$c_{ij}u_j(\xi,t)=\sum_{m=1}^{EB}\left\{\int_{\Gamma m}u_{ij}^{*}(\xi,X)\Phi d\Gamma\right\}p_j(X,t)-\sum_{m=1}^{EB}\left\{\int_{\Gamma m}p_{ij}^{*}(\xi,X)\Phi d\Gamma\right\}u_j(X,t)$$

$$-\rho\sum_{n=1}^{EV}\left\{\int_{\Omega n}u_{ij}^{*}(\xi,X)\Phi d\Omega\right\}\ddot{u}_j(X,t)+\sum_{n=1}^{EV}\left\{\int_{\Omega n}\varepsilon_{jki}^{*}(\xi,X)\Phi d\Omega\right\}\sigma_{jk}^{p}(X,t) \quad [1]$$

Boundary element discretization

$$c_{ij}u_j=\sum_{m=1}^{EB}\{G_{ij}\}p_j-\sum_{m=1}^{EB}\{H_{ij}\}u_j-\sum_{m=1}^{EV}\{M_{ij}\}\ddot{u}_j+\sum_{m=1}^{EV}\{Q_{ij}\}\sigma_{jk}^{p} \quad [2]$$

$$c_{ij}u_j = \sum_{m=1}^{EB}\{G_{ij}\}p_j - \sum_{m=1}^{EB}\{H_{ij}\}u_j - \sum_{m=1}^{EV}\{M_{ij}\}\ddot{u}_j + \sum_{m=1}^{EV}\{Q_{ij}\}\sigma_{jk}^{p} \quad [2]$$



[A(x)]{x}={b} ⇨ {x}=… [3]

❑ FINITE ELEMENT METHOD - FEM

3-D dynamic inelastic problems

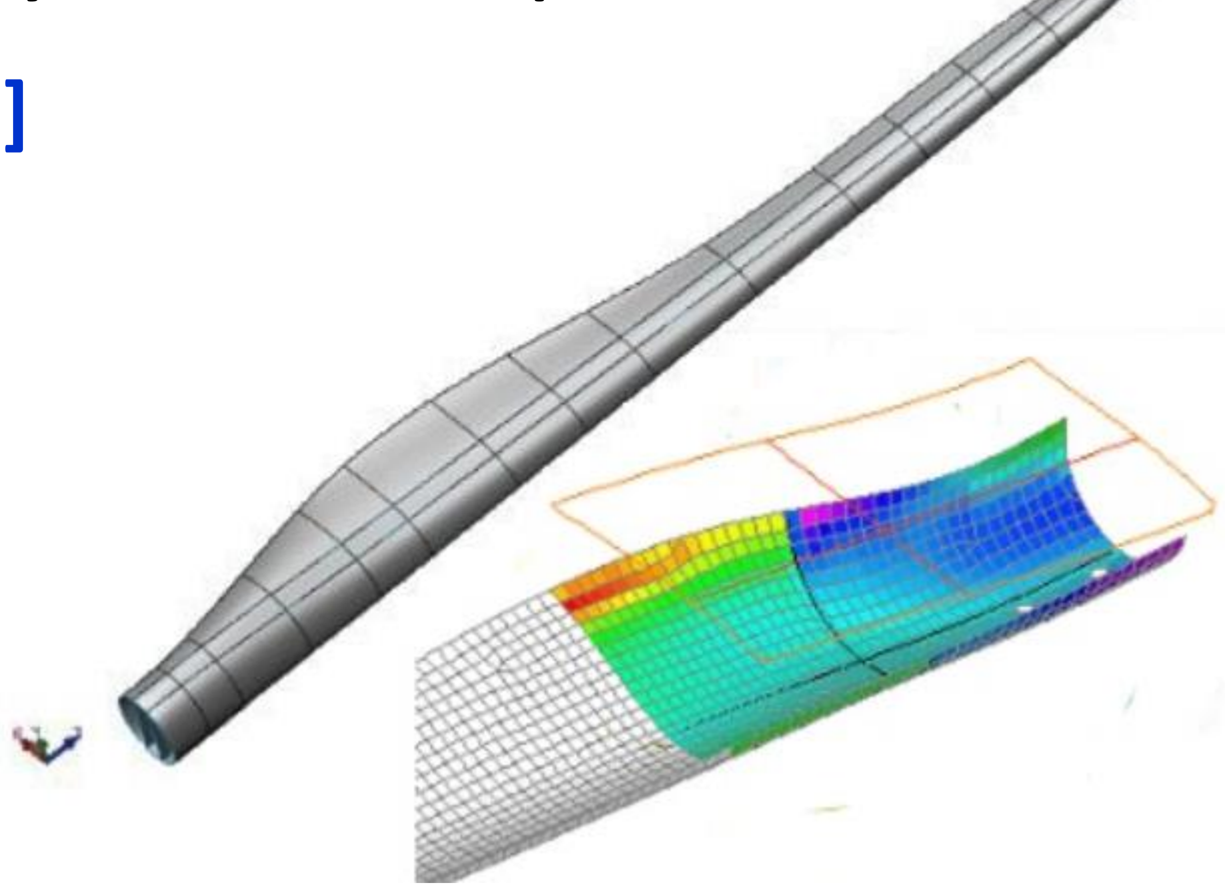
$$[M]\{\ddot{u}\}+[C]\{\dot{u}\}+\{F\}=\{R\} \quad [4]$$

⇩

- **Newmark**
- **Direct Substitution or others (e.g. Modified Newton-Raphson)**

⇩

[A(x)]{x}={b} ⇨ {x}=… [5]



BOUNDARY ELEMENT METHOD - BEM

[A(x)]{x}={b} ⇨ {x}=… [3]

$$[A] = \begin{vmatrix} F & R & Z & C & E & R & M & Z \\ P & B & S & G & P & J & S & K \\ G & A & J & B & L & Y & P & J \\ S & Y & K & Y & W & V & X & E \\ K & C & M & R & U & A & N & W \\ Z & L & W & P & L & M & Y & G \\ W & A & K & E & J & T & S & B \\ G & M & S & C & B & P & L & R \end{vmatrix}$$

[A] is a fully-populated and non-symmetric matrix with smaller size than that of FEM (dependent on the number of surface nodes)

FINITE ELEMENT METHOD - FEM

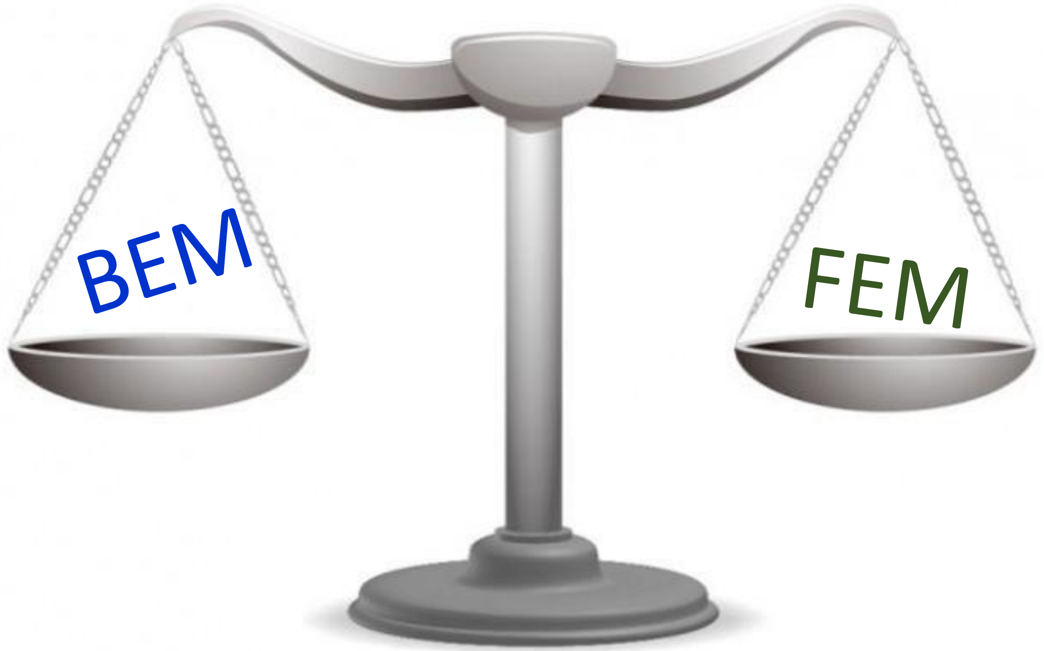
[A(x)]{x}={b} ⇨ {x}=… [5]

$$[A] = \begin{vmatrix} A & B & & & & & & & & & & & & & & & & & & \\ B & C & D & & & & & & & & & & & & & & & & & \\ & D & E & F & & & & & & & & & & & & & & & & \\ & & F & G & H & & & & & & & & & & & & & & & \\ & & & H & I & J & & & & & & & & & 0 & & & & & \\ & & & & J & K & L & & & & & & & & & & & & & \\ & & & & & L & M & N & & & & & & & & & & & & \\ & & & & & & N & O & P & & & & & & & & & & & \\ & & & & & & & P & Q & R & & & & & & & & & & \\ & & & & & & & & R & S & T & & & & & & & & & \\ & & & & & & & & & T & U & V & & & & & & & & \\ & & & & & & & & & & V & W & Y & & & & & & & \\ & & & & & & & & & & & Y & X & B & & & & & & \\ & & & & & & & & & & & & B & Z & X & & & & & \\ & & & 0 & & & & & & & & & & X & K & J & & & & \\ & & & & & & & & & & & & & & J & R & Q & & & \\ & & & & & & & & & & & & & & & Q & G & H & & \\ & & & & & & & & & & & & & & & & H & B & E & \\ & & & & & & & & & & & & & & & & & E & T & B \\ & & & & & & & & & & & & & & & & & & B & P \end{vmatrix}$$

[A] is a symmetric matrix with non-zero elements only along the principal diagonal (sparse) and bigger size than that of BEM (dependent on the number of finite element nodes)

CONCLUSIONS

WHICH METHOD APPEARS TO BE THE BEST FOR SOLVING NONLINEAR PROBLEMS?



AXIOM: The best method depends heavily on the nature of the problem being solved and the geometry involved.

Thank you for your attention!

Questions? (We love ‘em)